Symmetric Chain Decomposition of Necklace Posets

Vivek Dhand

October 4, 2011

Theorem (Sperner, 1928)

If $S_1, \ldots, S_k \subset \{1, 2, \ldots, n\}$ *are mutually incomparable subsets, then:*

$$k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Theorem (Sperner, 1928)

If $S_1, \ldots, S_k \subset \{1, 2, \ldots, n\}$ *are mutually incomparable subsets, then:*

$$k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We can rephrase this result in terms of the Boolean lattice $B(n) =$ the lattice of subsets of $\{1, 2, \ldots, n\}$.

Theorem (Sperner, 1928)

If $S_1, \ldots, S_k \subset \{1, 2, \ldots, n\}$ *are mutually incomparable subsets, then:*

$$k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We can rephrase this result in terms of the Boolean lattice $B(n) =$ the lattice of subsets of $\{1, 2, \ldots, n\}$.

Definition:

An **anti-chain** in a poset is a set of mutually incomparable elements. Note that a given rank of a ranked poset must be an anti-chain.

Theorem (Sperner, 1928)

If $S_1, \ldots, S_k \subset \{1, 2, \ldots, n\}$ are mutually incomparable subsets, then:

$$k \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We can rephrase this result in terms of the Boolean lattice $B(n) =$ the lattice of subsets of $\{1, 2, \ldots, n\}$.

Definition:

An **anti-chain** in a poset is a set of mutually incomparable elements. Note that a given rank of a ranked poset must be an anti-chain.

For $B(n)$, Sperner's theorem says that the cardinality of any anti-chain must be less than or equal to the cardinality of the middle rank(s). This condition is called the **Sperner property**.

Let P be a finite ranked poset of length n. Let p_i denote the cardinality of the i-th rank of P. Let $V(P)$ denote the $\mathbb{C}$-vector space with basis given by the elements of P. The following result was inspired by Richard Stanley's work on Weyl groups and the Hard Lefschetz Theorem.

Let P be a finite ranked poset of length n. Let p_i denote the cardinality of the i-th rank of P. Let $V(P)$ denote the $\mathbb{C}$-vector space with basis given by the elements of P. The following result was inspired by Richard Stanley's work on Weyl groups and the Hard Lefschetz Theorem.

Theorem (Proctor, 1982)

$V(P)$ carries a representation of $\mathfrak{sl}_2\mathbb{C}$ if and only if:

1. *P is rank-symmetric: $p_i = p_{n-i}$ for all i.*
2. *P is unimodal: $p_0 \leq p_1 \cdots \leq p_{\text{middle}} \geq \cdots \geq p_n$.*
3. *P is strongly Sperner: for each k, the middle k ranks form a maximal k-family of anti-chains.*

Let P be a finite ranked poset of length n. Let p_i denote the cardinality of the i-th rank of P. Let $V(P)$ denote the $\mathbb{C}$-vector space with basis given by the elements of P. The following result was inspired by Richard Stanley's work on Weyl groups and the Hard Lefschetz Theorem.

Theorem (Proctor, 1982)

$V(P)$ carries a representation of $\mathfrak{sl}_2\mathbb{C}$ if and only if:

1. *P is rank-symmetric: $p_i = p_{n-i}$ for all i.*
2. *P is unimodal: $p_0 \leq p_1 \cdots \leq p_{\text{middle}} \geq \cdots \geq p_n$.*
3. *P is strongly Sperner: for each k, the middle k ranks form a maximal k-family of anti-chains.*

Note that Proctor's Theorem implies Sperner's Theorem because $B(n) \simeq \{0,1\}^n$ and therefore $V(B(n)) \simeq V_1^{\otimes n}$, where V_1 denotes the standard representation of $\mathfrak{sl}_2\mathbb{C}$.

Definition:

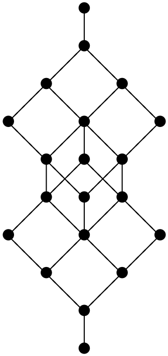
A finite ranked poset is called a **symmetric chain order** if it has a decomposition into rank-symmetric saturated chains.

Definition:

A finite ranked poset is called a **symmetric chain order** if it has a decomposition into rank-symmetric saturated chains.

Example:

$L(3,3)$

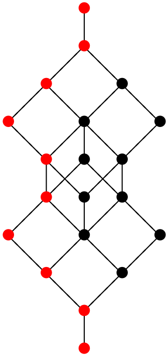


Definition:

A finite ranked poset is called a **symmetric chain order** if it has a decomposition into rank-symmetric saturated chains.

Example:

$L(3,3)$

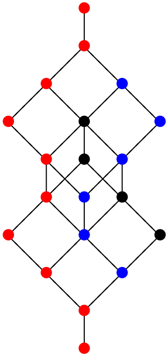


Definition:

A finite ranked poset is called a **symmetric chain order** if it has a decomposition into rank-symmetric saturated chains.

Example:

$L(3,3)$

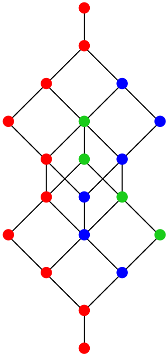


Definition:

A finite ranked poset is called a **symmetric chain order** if it has a decomposition into rank-symmetric saturated chains.

Example:

$L(3,3)$



It is not too hard to see that a symmetric chain order is rank-symmetric, unimodal, and strongly Sperner.

It is not too hard to see that a symmetric chain order is rank-symmetric, unimodal, and strongly Sperner.

On the other hand, there are examples of $\mathfrak{sl}_2\mathbb{C}$-posets which are not symmetric chain orders.

It is not too hard to see that a symmetric chain order is rank-symmetric, unimodal, and strongly Sperner.

On the other hand, there are examples of $\mathfrak{sl}_2\mathbb{C}$-posets which are not symmetric chain orders.

It turns out that symmetric chain decomposition is related to the representation theory of the *quantum group* $U_q(\mathfrak{sl}_2\mathbb{C})$.

If V is a finite-dimensional $U_q(\mathfrak{sl}_2\mathbb{C})$-representation, then the crystal graph Γ_V of V is a union of symmetric chains.

If V is a finite-dimensional $U_q(\mathfrak{sl}_2\mathbb{C})$-representation, then the crystal graph Γ_V of V is a union of symmetric chains.

If V and W are $U_q(\mathfrak{sl}_2\mathbb{C})$-representations, then the crystal graph of $V \otimes W$ yields a symmetric chain decomposition of $\Gamma_V \times \Gamma_W$.

If V is a finite-dimensional $U_q(\mathfrak{sl}_2\mathbb{C})$-representation, then the crystal graph Γ_V of V is a union of symmetric chains.

If V and W are $U_q(\mathfrak{sl}_2\mathbb{C})$-representations, then the crystal graph of $V \otimes W$ yields a symmetric chain decomposition of $\Gamma_V \times \Gamma_W$.

For example, if V_1 denotes the two-dimensional irreducible representation of $U_q(\mathfrak{sl}_2\mathbb{C})$, then the crystal graph of $V_1^{\otimes n}$ gives a symmetric chain decomposition of $B(n)$.

If V is a finite-dimensional $U_q(\mathfrak{sl}_2\mathbb{C})$-representation, then the crystal graph Γ_V of V is a union of symmetric chains.

If V and W are $U_q(\mathfrak{sl}_2\mathbb{C})$-representations, then the crystal graph of $V \otimes W$ yields a symmetric chain decomposition of $\Gamma_V \times \Gamma_W$.

For example, if V_1 denotes the two-dimensional irreducible representation of $U_q(\mathfrak{sl}_2\mathbb{C})$, then the crystal graph of $V_1^{\otimes n}$ gives a symmetric chain decomposition of $B(n)$.

This crystal graph SCD is equivalent to the well-known Greene-Kleitman SCD of $B(n)$.

If V is a finite-dimensional $U_q(\mathfrak{sl}_2\mathbb{C})$-representation, then the crystal graph Γ_V of V is a union of symmetric chains.

If V and W are $U_q(\mathfrak{sl}_2\mathbb{C})$-representations, then the crystal graph of $V \otimes W$ yields a symmetric chain decomposition of $\Gamma_V \times \Gamma_W$.

For example, if V_1 denotes the two-dimensional irreducible representation of $U_q(\mathfrak{sl}_2\mathbb{C})$, then the crystal graph of $V_1^{\otimes n}$ gives a symmetric chain decomposition of $B(n)$.

This crystal graph SCD is equivalent to the well-known Greene-Kleitman SCD of $B(n)$.

While these explicit constructions are nice, it should be pointed out how easy it is to show by induction that a product of symmetric chain orders is a symmetric chain order.

Definition:

Let X be a set. Define the set of n-bead **necklaces** with labels in X to be:

$$X^n/\mathbb{Z}_n$$

where $\mathbb{Z}_n$ acts by cyclic permutation of the factors.

Definition:

Let X be a set. Define the set of n-bead **necklaces** with labels in X to be:

$$X^n/\mathbb{Z}_n$$

where $\mathbb{Z}_n$ acts by cyclic permutation of the factors.

Notation:

Given a sequence $(x_1, \ldots, x_n) \in X^n$, let $[x_1, \ldots, x_n]$ denote the corresponding necklace in $X^n/\mathbb{Z}_n$.

Definition:

Let X be a set. Define the set of n-bead **necklaces** with labels in X to be:

$$X^n/\mathbb{Z}_n$$

where $\mathbb{Z}_n$ acts by cyclic permutation of the factors.

Notation:

Given a sequence $(x_1, \ldots, x_n) \in X^n$, let $[x_1, \ldots, x_n]$ denote the corresponding necklace in $X^n/\mathbb{Z}_n$.

Note that if P is a ranked poset, then $P^n/\mathbb{Z}_n$ is also a ranked poset, where $[a] < [b]$ if there exists $x \in [a], y \in [b]$ such that $x < y$.

Definition:

Let X be a set. Define the set of n-bead **necklaces** with labels in X to be:

$$X^n/\mathbb{Z}_n$$

where $\mathbb{Z}_n$ acts by cyclic permutation of the factors.

Notation:

Given a sequence $(x_1, \ldots, x_n) \in X^n$, let $[x_1, \ldots, x_n]$ denote the corresponding necklace in $X^n/\mathbb{Z}_n$.

Note that if P is a ranked poset, then $P^n/\mathbb{Z}_n$ is also a ranked poset, where $[a] < [b]$ if there exists $x \in [a], y \in [b]$ such that $x < y$.

Example:

$B(n)/\mathbb{Z}_n \simeq \{0,1\}^n/\mathbb{Z}_n$ is called the poset of n-bead **binary necklaces**.

Theorem (Griggs-Killian-Savage, 2004)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order if n is prime.

Theorem (Griggs-Killian-Savage, 2004)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order if n is prime.

Theorem (Jordan, 2010)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order for all n.

Theorem (Griggs-Killian-Savage, 2004)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order if n is prime.

Theorem (Jordan, 2010)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order for all n.

Recently, two new and independent proofs of this result have been obtained by Duffus-McKibben-Sanders-Thayer and Hersh-Schilling.

Theorem (Griggs-Killian-Savage, 2004)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order if n is prime.

Theorem (Jordan, 2010)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order for all n.

Recently, two new and independent proofs of this result have been obtained by Duffus-McKibben-Sanders-Thayer and Hersh-Schilling.

Theorem (Dhand, 2011)

If P is a symmetric chain order, then so is $P^n/\mathbb{Z}_n$.

Theorem (Griggs-Killian-Savage, 2004)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order if n is prime.

Theorem (Jordan, 2010)

$B(n)/\mathbb{Z}_n$ is a symmetric chain order for all n.

Recently, two new and independent proofs of this result have been obtained by Duffus-McKibben-Sanders-Thayer and Hersh-Schilling.

Theorem (Dhand, 2011)

If P is a symmetric chain order, then so is $P^n/\mathbb{Z}_n$.

In particular, if P is a single chain with k vertices, then the poset of n-bead k-ary necklaces is a symmetric chain order.

Sketch of the proof:

Let's first consider the case of binary necklaces.

We can encode the data of an n-bead binary necklace in a slightly more efficient way by counting the number of steps between consecutive "ones" (reading clockwise, let's say).

Sketch of the proof:

Let's first consider the case of binary necklaces.

We can encode the data of an n-bead binary necklace in a slightly more efficient way by counting the number of steps between consecutive "ones" (reading clockwise, let's say).

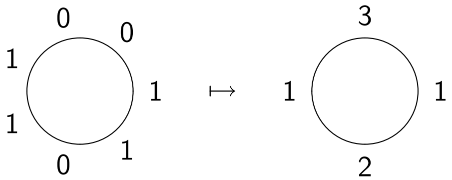
In this way, we obtain a partition of n which is ordered up to cyclic permutation, i.e. a **partition necklace**.

Sketch of the proof:

Let's first consider the case of binary necklaces.

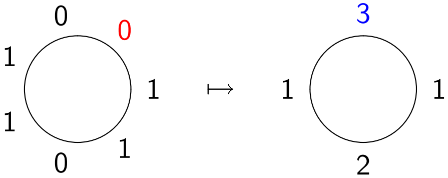
We can encode the data of an n-bead binary necklace in a slightly more efficient way by counting the number of steps between consecutive "ones" (reading clockwise, let's say).

In this way, we obtain a partition of n which is ordered up to cyclic permutation, i.e. a **partition necklace**.

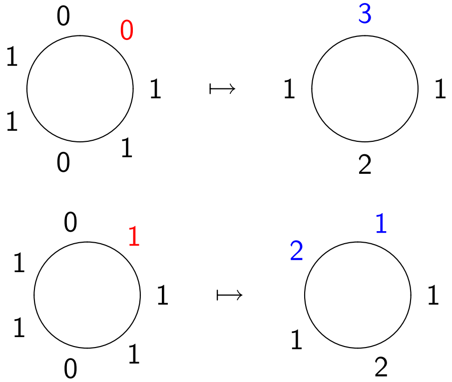


Moreover, changing a “zero” to a “one” in a binary necklace corresponds to *refinement* of partitions:

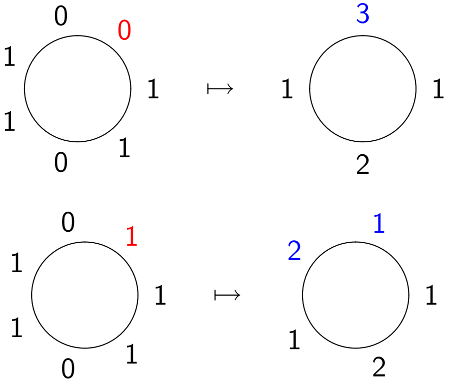
Moreover, changing a "zero" to a "one" in a binary necklace corresponds to *refinement* of partitions:



Moreover, changing a “zero” to a “one” in a binary necklace corresponds to *refinement* of partitions:



Moreover, changing a "zero" to a "one" in a binary necklace corresponds to *refinement* of partitions:



Therefore, $B(n)/\mathbb{Z}_n$ is isomorphic to the poset $P(n)$ of partition necklaces whose entries add up to n.

Example:

$B(6)/\mathbb{Z}_6$

$[0,0,0,0,0,0]$

$[1,0,0,0,0,0]$

$[1,1,0,0,0,0]$ $[1,0,0,1,0,0]$ $[1,0,1,0,0,0]$

$[1,1,1,0,0,0]$ $[1,1,0,1,0,0]$ $[1,1,0,0,1,0]$ $[1,0,1,0,1,0]$

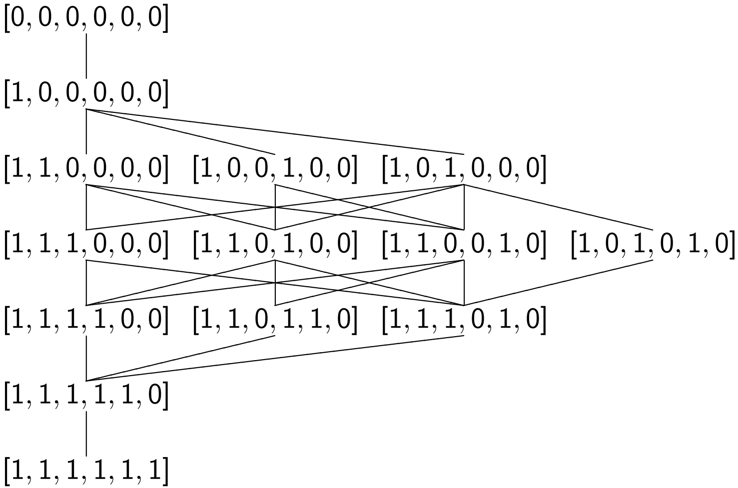
$[1,1,1,1,0,0]$ $[1,1,0,1,1,0]$ $[1,1,1,0,1,0]$

$[1,1,1,1,1,0]$

$[1,1,1,1,1,1]$

Example:

$B(6)/\mathbb{Z}_6$



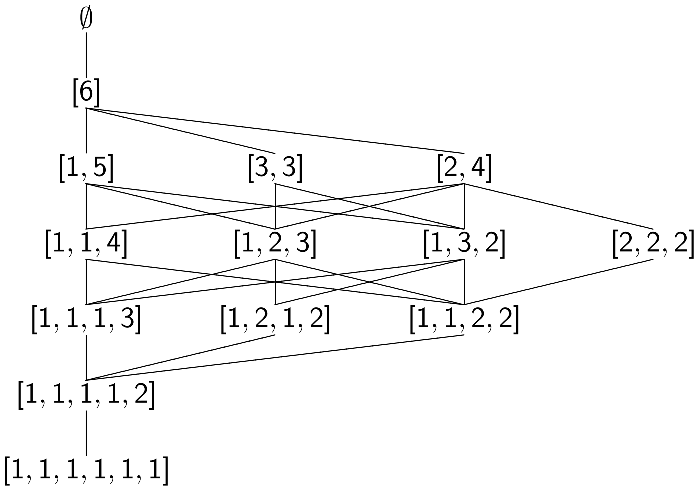
Example:

$P(6)$

$\emptyset$			
$[6]$			
$[1,5]$	$[3,3]$	$[2,4]$	
$[1,1,4]$	$[1,2,3]$	$[1,3,2]$	$[2,2,2]$
$[1,1,1,3]$	$[1,2,1,2]$	$[1,1,2,2]$	
$[1,1,1,1,2]$			
$[1,1,1,1,1,1]$			

Example:

$P(6)$



$P(n)$ has a natural decomposition into centered subposets with a remarkable factorization property.

$P(n)$ has a natural decomposition into centered subposets with a remarkable factorization property.

We call a partition necklace **fundamental** if each of its parts is greater than or equal to 2.

$P(n)$ has a natural decomposition into centered subposets with a remarkable factorization property.

We call a partition necklace **fundamental** if each of its parts is greater than or equal to 2.

We can associate a fundamental partition necklace to each partition necklace by adding each "one" to the first integer greater than one it meets as you read clockwise.

$P(n)$ has a natural decomposition into centered subposets with a remarkable factorization property.

We call a partition necklace **fundamental** if each of its parts is greater than or equal to 2.

We can associate a fundamental partition necklace to each partition necklace by adding each “one” to the first integer greater than one it meets as you read clockwise.

For example:

$$[1, 1, 2, 3, 1, 5, 8, 1] \mapsto [5, 3, 6, 8]$$

Let $Q_{[a_1,\ldots,a_r]}$ be the poset associated to the fundamental partition necklace $[a_1, \ldots, a_r]$ via the above map. This construction is equivalent to the **block code** decomposition of $B(n)/\mathbb{Z}_n$.

Let $Q_{[a_1,\ldots,a_r]}$ be the poset associated to the fundamental partition necklace $[a_1, \ldots, a_r]$ via the above map. This construction is equivalent to the **block code** decomposition of $B(n)/\mathbb{Z}_n$.

Note: we will exclude the partition necklaces $\emptyset$ and $[1, 1, \ldots, 1]$ from $Q_{[n]}$ in order to make the following result work:

Let $Q_{[a_1,\ldots,a_r]}$ be the poset associated to the fundamental partition necklace $[a_1,\ldots,a_r]$ via the above map. This construction is equivalent to the **block code** decomposition of $B(n)/\mathbb{Z}_n$.

Note: we will exclude the partition necklaces $\emptyset$ and $[1,1,\ldots,1]$ from $Q_{[n]}$ in order to make the following result work:

Lemma

If $[a_1,\ldots,a_r]$ *is aperiodic, then:*

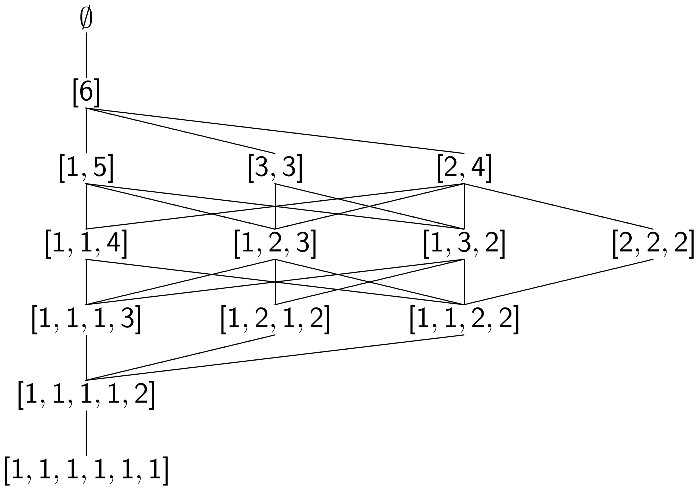
$$Q_{[a_1,\ldots,a_r]} \simeq Q_{[a_1]} \times \cdots \times Q_{[a_r]}$$

On the other hand, if $[a_1,\ldots,a_r] = [b,\ldots,b]$ *is periodic of period* d*, then:*

$$Q_{[a_1,\ldots,a_r]} \simeq Q_{[b]}^{\frac{r}{d}}/\mathbb{Z}_{\frac{r}{d}}$$

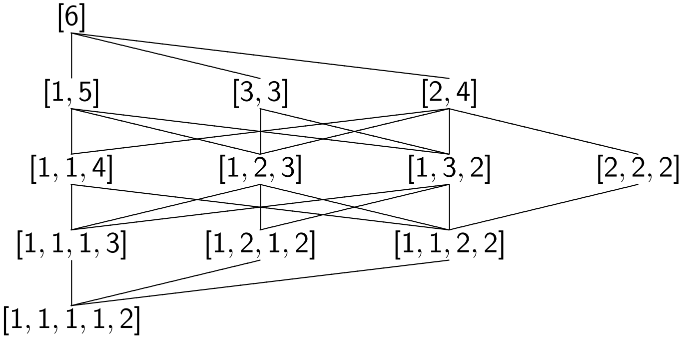
Example:

$P(6)$



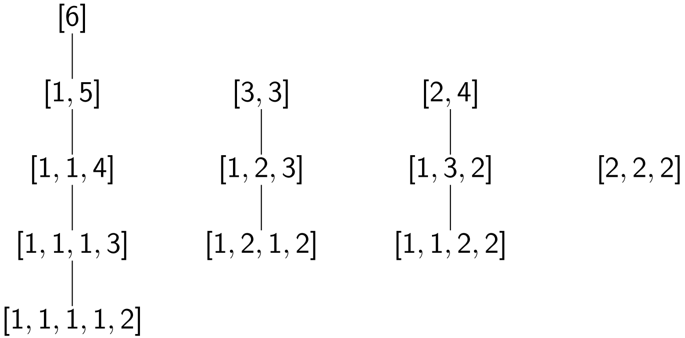
Example:

$P(6)$



Example:

$P(6)$



$Q_{[6]}$ $\qquad Q^2_{[3]}/\mathbb{Z}_2$ $\qquad Q_{[2]} \times Q_{[4]}$ $\qquad Q^3_{[2]}/\mathbb{Z}_3$

We can also associate a partition necklace to each n-bead $(k+1)$-ary necklace:

$$\{0, 1, \ldots, k\}^n/\mathbb{Z}_n \to B(kn)/\mathbb{Z}_{kn} \simeq P(kn)$$
$$j \mapsto 1^j 0^{k-j}$$

We can also associate a partition necklace to each n-bead $(k+1)$-ary necklace:

$$\{0, 1, \dots, k\}^n / \mathbb{Z}_n \to B(kn)/\mathbb{Z}_{kn} \simeq P(kn)$$
$$j \mapsto 1^j 0^{k-j}$$

For example:

$$\{0, 1, 2, 3, 4\}^3 / \mathbb{Z}_3 \to B(12)/\mathbb{Z}_{12} \to P(12)$$

$$[2, 1, 4] \mapsto [1, 1, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1] \mapsto [3, 4, 1, 1, 1, 1, 1] \in Q_{[8,4]}$$

We can also associate a partition necklace to each n-bead $(k+1)$-ary necklace:

$$\{0,1,\ldots,k\}^n/\mathbb{Z}_n \to B(kn)/\mathbb{Z}_{kn} \simeq P(kn)$$
$$j \mapsto 1^j 0^{k-j}$$

For example:

$$\{0,1,2,3,4\}^3/\mathbb{Z}_3 \to B(12)/\mathbb{Z}_{12} \to P(12)$$

$$[2,1,4] \mapsto [1,1,0,0,1,0,0,0,1,1,1,1] \mapsto [3,4,1,1,1,1,1] \in Q_{[8,4]}$$

Indeed, the image of this embedding is exactly the union of those $Q_{[a_1,\ldots,a_r]}$ where each a_i is divisible by k.

Let P be a symmetric chain order. It is straightforward to reduce the problem of $P^n/\mathbb{Z}_n$ to the case where P is a single chain.

Let P be a symmetric chain order. It is straightforward to reduce the problem of $P^n/\mathbb{Z}_n$ to the case where P is a single chain.

Moreover, by the above results, the case of a single chain reduces to $B(n)/\mathbb{Z}_n$.

Let P be a symmetric chain order. It is straightforward to reduce the problem of $P^n/\mathbb{Z}_n$ to the case where P is a single chain.

Moreover, by the above results, the case of a single chain reduces to $B(n)/\mathbb{Z}_n$.

Now by the factorization property of the Q-posets, we can return to the case of $P^d/\mathbb{Z}_d$, where $d < n$, and apply induction.

Further Directions:

Can the above theorem be proved using the representation theory of $U_q(\mathfrak{sl}_2\mathbb{C})$?

Further Directions:

Can the above theorem be proved using the representation theory of $U_q(\mathfrak{sl}_2\mathbb{C})$?

Unfortunately, $U_q(\mathfrak{sl}_2\mathbb{C})$ does not act on $V^{\otimes n}/\mathbb{Z}_n$ in any obvious way.

Further Directions:

Can the above theorem be proved using the representation theory of $U_q(\mathfrak{sl}_2\mathbb{C})$?

Unfortunately, $U_q(\mathfrak{sl}_2\mathbb{C})$ does not act on $V^{\otimes n}/\mathbb{Z}_n$ in any obvious way.

However, the fact that $Q_{[a_1,\ldots,a_r]}$ is ultimately a disjoint union of products of symmetric chains implies that there still might be some way to make this work.

Further Directions:

If P is a single chain with $m+1$ vertices and $G = \Sigma_n$, then P^n/G is isomorphic to $L(m, n)$.

Further Directions:

If P is a single chain with $m+1$ vertices and $G = \Sigma_n$, then P^n/G is isomorphic to $L(m,n)$.

$L(m,n)$ is the poset of unordered partitions with n parts where each part is at most m, partially ordered by inclusion of their Young diagrams. The SCD question for this poset is Stanley's famous $L(m,n)$ conjecture.

Further Directions:

If P is a single chain with $m+1$ vertices and $G = \Sigma_n$, then P^n/G is isomorphic to $L(m,n)$.

$L(m,n)$ is the poset of unordered partitions with n parts where each part is at most m, partially ordered by inclusion of their Young diagrams. The SCD question for this poset is Stanley's famous $L(m,n)$ conjecture.

We have a "tropical" decomposition of $L(m,n)$ into centered sub-posets $Q(m,n,e_1,\ldots,e_k)$, but these posets are still very complicated. However, we can show for $n, e_1, \ldots, e_k$ fixed and sufficiently large m that $Q(m,n,e_1,\ldots,e_k)$ has SCD. In other words, "most" of $L(m,n)$ is a symmetric chain order (joint work in progress with P. Magyar).

References

J. Griggs, C. Killian, C. Savage (2004)
Venn diagrams and symmetric chain decompositions in the Boolean lattice.
Electron. J. Combin. 11.

K. Jordan (2010)
The necklace poset is a symmetric chain order.
J. Combin. Theory Ser. A 117, No. 6, 625641.

R. A. Proctor (1982)
Representations of $\mathfrak{sl}_2\mathbb{C}$ on posets and the Sperner property.
Siam. J. Alg. Disc. Math. Vol. 3, No. 2.

E. Sperner(1928)
Ein Satz ber Untermengen einer endlichen Menge.
Mathematische Zeitschrift **27** (1), 544548.

The End.

Thanks for listening!