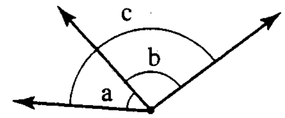
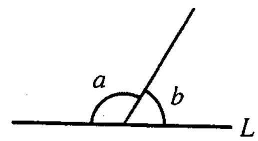
Background Knowledge

Here is a list of the geometric facts at our disposal at this point. These will be used in the examples and homework problems in this chapter. Several additional facts will be added to this list in Section 4.2.

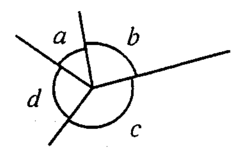
• The measures of adjacent angles add.
($c = a + b$.)
Abbreviation: **∠s add.**



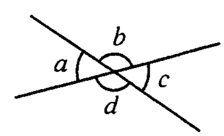
• The sum of adjacent angles on a straight line is 180°.
(If L is a line then $a + b = 180°$.)
Abbreviation: **∠s on a line.**



• The sum of adjacent angles around a point is 360°.
($a + b + c + d = 360°$.)
Abbreviation: **∠s at a pt.**

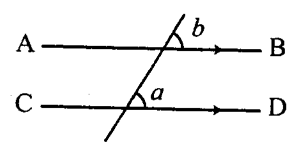


• Vertically opposite angles are equal.
(At the intersection of two straight lines, $a = c$ and $b = d$).
Abbreviation: **vert. ∠s.**



• When a transversal intersects parallel lines, corresponding angles are equal.
(If $\overline{AB} \parallel \overline{CD}$ then $a = b$.)
Abbreviation: **corr. ∠s, $\overline{AB} \parallel \overline{CD}$.**

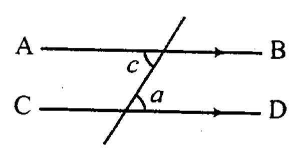
• Conversely, if $a = b$ then $\overline{AB} \parallel \overline{CD}$.
Abbreviation: **corr. ∠s converse.**



• When a transversal intersects parallel lines, alternate interior angles are equal.

(If $\overline{AB} \parallel \overline{CD}$ then $a = c$.)
Abbreviation: **alt. ∠s, $\overline{AB} \parallel \overline{CD}$.**

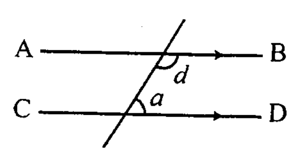
• Conversely, if $a = c$ then $\overline{AB} \parallel \overline{CD}$.
Abbreviation: **alt. ∠s converse.**



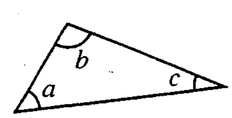
• When a transversal intersects parallel lines, interior angles on the same side of the transversal are supplementary.

(If $\overline{AB} \parallel \overline{CD}$ then $a + d = 180°$.)
Abbreviation: **int. ∠s, $\overline{AB} \parallel \overline{CD}$.**

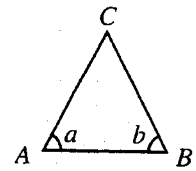
• Conversely, if $a + d = 180$ then $\overline{AB} \parallel \overline{CD}$.
Abbreviation: **int. ∠s converse.**



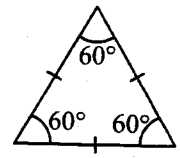
• The angle sum of any triangle is 180°. (*)
($a + b + c = 180°$.)
Abbreviation: **∠ sum of Δ.**



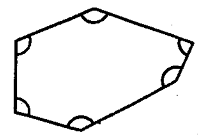
• Base angles of an isosceles triangle are equal. (*)
(If $AC = BC$ then $a = b$.)
Abbreviation: **base ∠s of isos. Δ.**



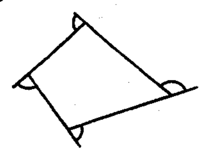
• Each interior angle of an equilateral triangle is 60°. (*)
Abbreviation: **equilat. Δ.**



• The sum of the interior angles of an n-gon is $180(n - 2)$.
Abbreviation: **∠ sum of n-gon.**



• The sum of the exterior angles of a convex n-gon is 360.
Abbreviation: **ext. ∠s of cx. n-gon.**

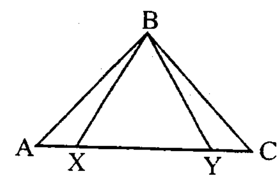


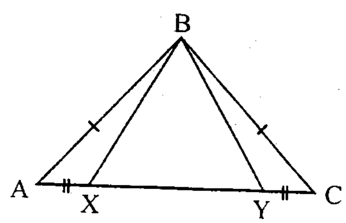
Reprinted from pp. 71-73 of
Parker, T. H. & Baldridge, S. J. (2007) Elementary Geometry for Teachers. Quebecor, MI.

4.3 Applying Congruences

Once students have learned the triangle tests, they can move beyond unknown angles problems to the next stage in the curriculum: *proving* facts about side-lengths and angles within figures. This section is an introduction to the simplest proofs. At this point, students appreciate and perhaps enjoy geometry puzzles and short proofs. They are prepared for high school geometry.

EXAMPLE 3.1. *Let $\triangle ABC$ be an isosceles triangle with $AB = BC$. Let X and Y be distinct points on $\overline{AC}$ such that $AX = YC$. Prove that $\triangle XBY$ is also isosceles.*





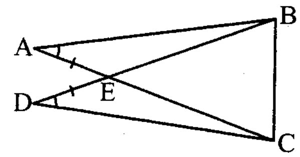
Given: $AB = BC$ and $AX = YC$.
To Prove: $BX = BY$.

Proof.

	$AB = BC$	given
	$\angle A = \angle C$	base $\angle$s of isos. Δ
	$AX = YC$	given
	$\therefore \Delta AXB \cong \Delta CYB$	SAS.
$\therefore$	$BX = BY$	corr. sides of $\cong$ Δs.

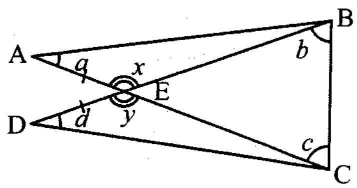
The triangle tests can also be used in unknown angle proofs.

EXAMPLE 3.2. *In the figure, ∠A = ∠D and AE = DE. Prove ∠ECB = ∠EBC.*



To prove this, mark angles as shown and concentrate your attention on the two shaded triangles.

Given: $a = d$ and $AE = DE$.
To Prove: $b = c$.



Proof.	$a = d$	given
	$AE = DE$	given
	$x = y$	vert. ∠s
	$\therefore \triangle AEB \cong \triangle DEC$	ASA.
	$\therefore EB = EC$	corr. sides of ≅ Δs.
$\therefore$	$b = c$	base ∠s of isos. Δ.

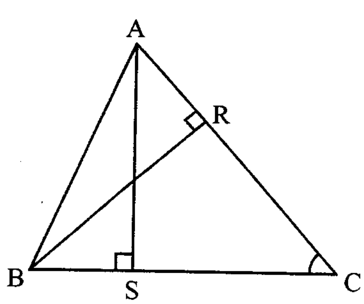
Study Examples 3.1 and 3.2 for a moment. In each proof, Line 4 states that two triangles are congruent by a congruence test, Lines 1-3 are the facts needed to apply the test, and Line 5 is a conclusion based on this congruence. In this chapter, almost all of the proofs will have this format. It is simple, but has many applications!

false starts in proofs

As students learn to construct such proofs, it is easy for them to make a "false start" — their first approach doesn't work. *This is completely normal!* Proofs are like puzzles: the fun lies in trying different strategies to find a solution. The reward, like the reward in solving a tricky puzzle, is a feeling of accomplishment from succeeding at something that you know is not easy. In fact, geometric proofs were a common amusement of educated people in the 19th century, just as crossword and sudoku puzzles are today.

The next example is a problem in which it is easy to make a false start. The figure contains several pairs of congruent triangles; which pair should be used in the proof? Try to find a strategy before you look at the proof written below. Here is a trick that helps: color, highlight or shade the segments that appear in the "Given" and the "To Prove" statements. *Look for a pair of congruent triangles that contain these sides.*

EXAMPLE 3.3. *In the figure, CA = CB. Prove that AS = BR.*



Given: $CA = CB$.
To Prove: $AS = BR$.

Proof.	In $\triangle ACS$ and $\triangle BCR$,	
	$CA = CB$	given
	$\angle C = \angle C$	common angle
	$\angle R = \angle S = 90°$	given
	$\therefore \triangle ACS \cong \triangle BCR$	AAS.
$\therefore$	$AS = BR$	corr. sides of ≅ Δs.

EXERCISE 3.4. *This proof used the congruence $\Delta ACS \cong \Delta BCR$. Name (without proof) two other pairs of congruent triangles in this figure (first label the intersection point in the middle of the figure T).*

These examples indicate that there are two levels of congruent triangle proofs: ones that are especially simple because the figure contains only one pair of congruent triangles, and ones in which the student must find the appropriate congruent triangles from among several pairs. Careful textbooks (and teachers!) provide plenty of practice at the first level before challenging students with problems at the second level. The proofs in your homework set (HW Set 14) are similar to a well-written eighth grade textbook. As you do those proofs, notice how they slowly increase in difficulty.

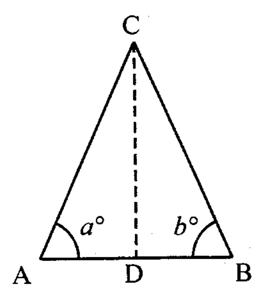
Geometric proofs for cutting and folding explanations

The congruence tests can be used to prove many of the facts that were introduced in fifth and sixth grade using cutting and folding methods. This section explains several such proofs and gives other applications of the congruence tests. As you will see, many of the elementary school "folding proofs" contain the key idea of a complete geometric proof.

THEOREM 3.5. *In an isosceles triangle, base angles are congruent.*

(Abbreviation: **base ∠s of isos. Δ.***)*

The Folding Proof. The proof below is just the detailed explanation of the fifth grade picture proof on page 62 of Primary Math 5B.



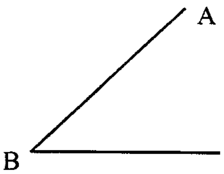
Given: ΔABC with $AC = BC$.
To prove: $a = b$.

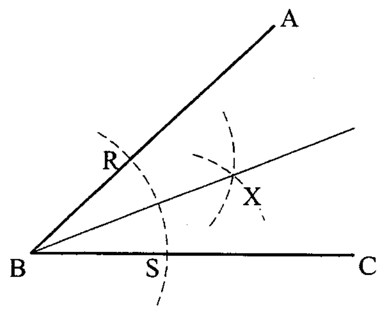
Construction: Let D be the midpoint of $\overline{AB}$.
Draw $\overline{CD}$.

Proof.	$AC = BC$	given
	$AD = DB$	D is the midpoint
	$DC = DC$	common.
	$\Delta ADC \cong \Delta BDC$	SSS
	$\therefore a = b$	corr. ∠s of ≅ Δs.

When we first introduced compass and straightedge constructions we focused on the steps required and only "paper-folding" arguments to verify that the construction accomplished its intended purpose. We can now replace those paper-folding arguments by proofs using congruent triangles. For example, the construction below shows how to bisect an angle.

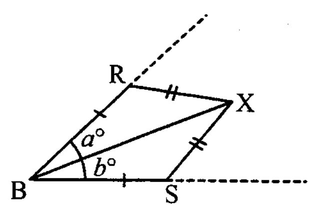
EXAMPLE 3.6. *Bisect $\angle ABC$.*
(Abbreviation: **bisect ∠ constr.***)*





- Draw circle, center B, any radius. Mark R and S where the circle intersects the sides of the angle.
- Draw circle, center R, radius RS. Draw circle, center S, radius RS.
- Mark X where these two circles intersect.
- Draw $\overrightarrow{BX}$.

To confirm that this construction works, we prove that $\angle ABX$ and $\angle CBX$ are equal. The paper-folding argument suggests showing that ΔBRX and ΔBSX are congruent.



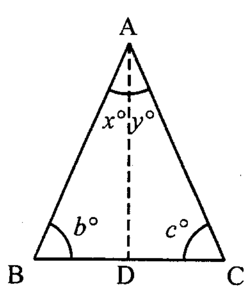
Given: $BR = BS$, $RX = SX$.
To prove: $a = b$.

Proof.	$BR = BS$	given
	$RX = SX$	given
	$BX = BX$	common
	$\Delta BRX \cong BSX$	SSS.
	$\therefore a = b$	corr. ∠s of ≅ Δs.

This bisection construction, and all of the constructions we did in Section 2.4, are *theorems* as well as constructions. Each entails a construction followed by a proof that the construction works. Therefore we can use the constructions as reasons in proofs, as in the proof below.

THEOREM 3.7. *If two angles of a triangle are congruent, then the triangle is isosceles.*

(Abbreviation: **base ∠s converse.***)*



Given: ΔABC with $b = c$.
To Prove: $AB = AC$.

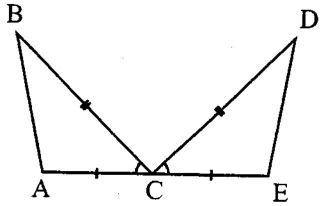
Construction: Bisect $\angle A$ and mark angles x and y as shown.

Proof.	$b = c$	given
	$x = y$	bisect ∠ const.
	$AD = AD$	common
	$\therefore \Delta BAD \cong \Delta CAD$	AAS.
$\therefore$	$AB = AC$	corr. sides of ≅ Δs.

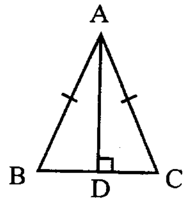
Homework Set 14

This homework set gives more practice with proofs using congruent triangles. Please write Elementary Proofs to the following problems. Most of these problem are similar to homework problems found in Japan and Hong Kong in eighth grade.

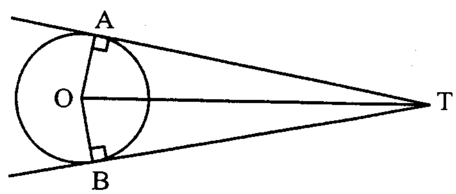
1. In the figure, prove that $\angle ABC = \angle EDC$.



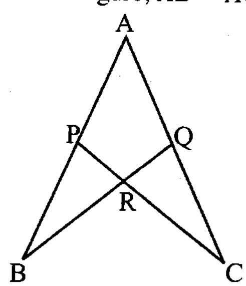
2. In the figure, prove that $BD = DC$.



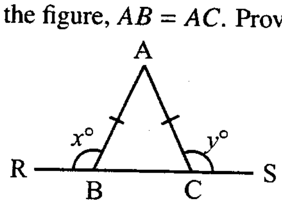
3. In the figure, O is the center of the circle. Prove that $TA = TB$.



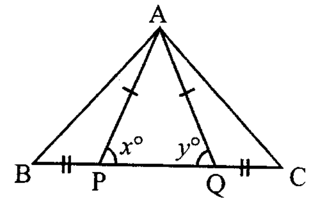
4. In the figure, $AB = AC$, $AP = AQ$. Prove that $BQ = CP$.



5. In the figure, $AB = AC$. Prove that $x = y$.



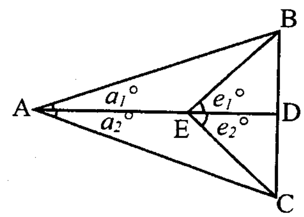
6. In the figure, $BP = QC$, $AP = AQ$, and $x = y$. Prove that $AB = AC$.



7. In the figure, $a_1 = a_2$, $e_1 = e_2$. Prove that

   a) $\Delta ABE \cong \Delta ACE$.

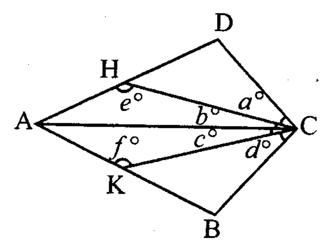
   b) $AB = AC$ and $\overline{AD} \perp \overline{BC}$.



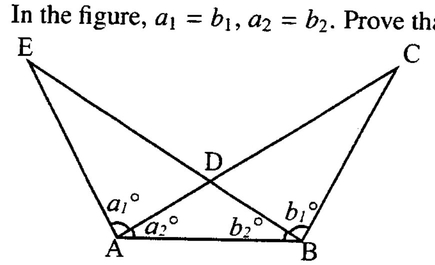
8. In the figure, $a = d$, $b = c$, and $CB = CD$. Prove that

   a) $\Delta ADC \cong \Delta ABC$.

   b) $e = f$.



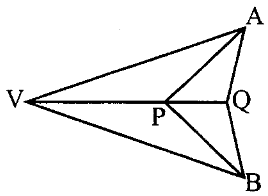
9. In the figure, $a_1 = b_1$, $a_2 = b_2$. Prove that $AC = BE$.



10. In the figure, $AV = BV$, $AP = BP$. Prove that

    a) $\angle APV = \angle BPV$.

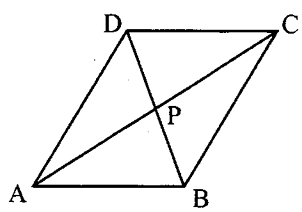
    b) $AQ = BQ$.



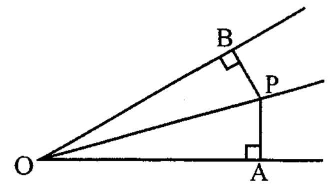
11. In the figure, $AB = CD$, $BC = AD$. Prove that

a) $\angle ADB = \angle DBC$.

b) $\overline{AC}$ and $\overline{BD}$ bisect each other.



12. In the figure, P is equidistant from the lines OA and OB (i.e. $PA = PB$). Prove that OP bisects $\angle AOB$.

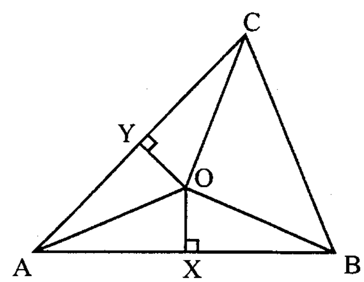


13. In the figure, OX and OY are the perpendicular bisectors of AB and AC respectively. Prove that

a) $\Delta OAX \cong \Delta OBX$.

b) $OA = OB = OC$.

c) What can you say about the circle with center O and radius OA? (Hint: Draw it in your book.)



4.4 Parallelograms, Rectangles, Rhombuses, and Kites

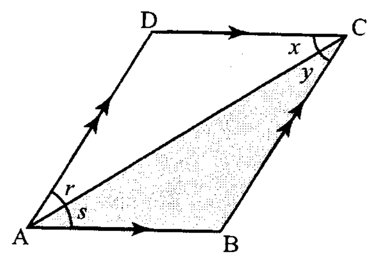
You have already seen how the grade 5 (page 68) and grade 6 (page 64–64) Primary Mathematics textbooks present facts about parallelograms. At this stage students can dig deeper and see that the facts they learned about parallelograms and other quadrilaterals are actually consequences of more fundamental facts, namely those listed as "Background Knowledge" at the beginning of this chapter. This section presents several such facts and gets you started on others. It ends with a review of the part of the K-8 curriculum leading to deductive geometry.

Parallelograms

By definition, a parallelogram is a quadrilateral in which both pairs of opposite sides are parallel. Using this definition and congruent triangles, we can prove additional properties of parallelograms as in the following proofs.

(1) In ΔABC and ΔCDA,

$s = x$	alt. $\angle$s, $\overline{AB} \parallel \overline{CD}$
$r = y$	alt. $\angle$s, $\overline{AD} \parallel \overline{BC}$
$AC = AC$	common
$\Delta ABC \cong \Delta CDA$	ASA
$\therefore AB = CD$ and $AD = BC$	corr. sides of $\cong$ Δs.



$\therefore$ **Opposite sides are equal.**

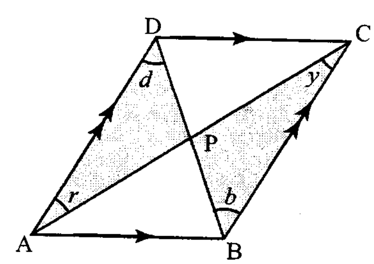
(2)

$\Delta ABC \cong \Delta CDA$	from part (1)
$\therefore \angle B = \angle D$	corr. $\angle$s of $\cong$ Δs
$\angle A = r + s$	$\angle$s add
$= x + y$	$s = x, r = y$
$= \angle C$	$\angle$s add.

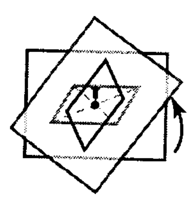
$\therefore$ **Opposite angles are equal.**

(3)

$\Delta ABC \cong \Delta CDA$	from part (1)
$r = y$	corr. $\angle$s of $\cong$ Δs
$AD = CB$	corr. sides of $\cong$ Δs
$b = d$	alt. $\angle$s, $\overline{AD} \parallel \overline{BC}$.
$\Delta ADP \cong \Delta CBP$	ASA
$\therefore AP = PC$ and $BP = PD$	corr. sides of $\cong$ Δs.



$\therefore$ **The diagonals bisect each other.**

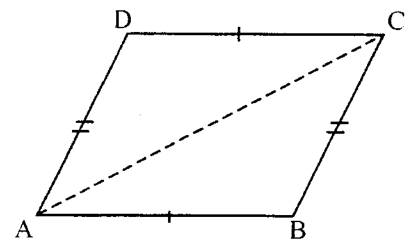


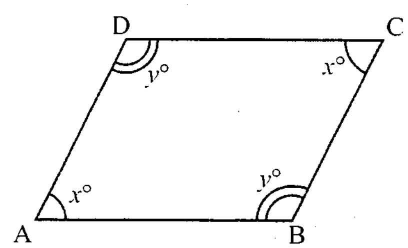
These three properties can also be seen by symmetry — but it is not a folding symmetry. Instead, put a pin at the center of parallelogram (the point where the diagonals intersect) and rotate 180° around that point. Do you see why this symmetry gives all three properties (opposite sides equal, opposite angles equal, diagonals bisect)?

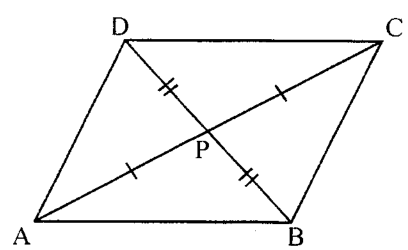
EXERCISE 4.1. *Try the activity suggested above using some tracing paper.*

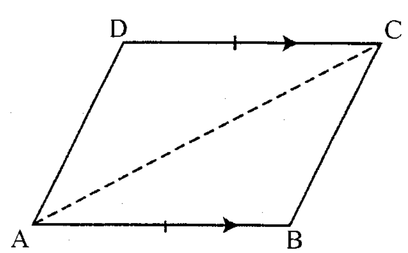
The converses of these three properties are also true: If a quadrilateral satisfies any one of the conditions (1), (2), (3) above, then that quadrilateral is a parallelogram. Students can be led to this conclusion by working through the following eighth-grade activity.

EXERCISE 4.2. *The chart below gives four different criteria for recognizing parallelograms. Fill in the blanks to complete the proofs.*

Both pairs of opposite sides are equal.

	$\Delta ABC \cong \Delta ACDA$ __________
	$\angle BAC = \angle DCA$ __________
	$\angle CAD = \angle ACB$ __________
	$\therefore \overline{AB} // \overline{CD}$ alt. $\angle$s converse
	$\therefore \overline{AD} // \overline{BC}$ alt. $\angle$s converse
	$\therefore ABCD$ is a parallelogram.

Both pairs of opposite angles are equal.

	$x + y + x + y =$ ___ $\angle$ sum n-gon
	$x + y = 180$
	$\therefore \overline{AB} // \overline{CD}$ int. $\angle$s converse.
	$\therefore \overline{AD} // \overline{BC}$ __________
	Is $ABCD$ is a parallelogram? __________

Diagonals bisect each other. 	$\Delta APB \cong \Delta CPD$ __________ $\angle PAB = \angle PCD$ __________ _____ // _____ __________ Similarly, $\Delta APD \cong \Delta CPB$ __________ $\angle ADP = \angle CBP$ __________ _____ // _____ __________ Is $ABCD$ is a parallelogram? __________

One pair of opposite sides are equal and parallel.

	$DC = AB$ given
	$\angle DCA = \angle BAC$ alt. $\angle$s, $\overline{AB} // \overline{DC}$
	$AC = AC$ common
	$\Delta ACD \cong \Delta CAB$ SAS
	$\angle DAC =$ _____ corr. $\angle$s, $\cong \Delta$
	$\overline{AD} // \overline{BC}$ __________
	Is $ABCD$ is a parallelogram? __________

Here are two quick applications of these criteria:

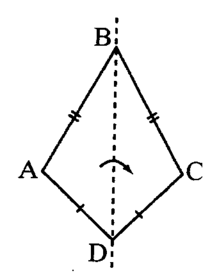
- A rhombus is a quadrilateral with all four sides equal in length; in particular, both pairs of opposite sides are equal. By the first criterion, *every rhombus is a parallelogram.*
- A rectangle is a quadrilateral with four right angles; in particular, both pairs of opposite angles of the rectangle are equal. By the second criterion, *every rectangle is a parallelogram.*

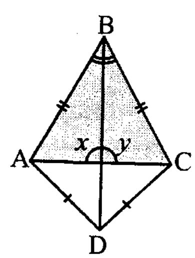
Kites

By definition, a kite is a quadrilateral in which two adjacent sides have equal length and the remaining two sides also have equal length. A kite has the following properties:

1. There is a pair of equal opposite angles.
2. One of its diagonals bisects a pair of opposite angles.
3. The two diagonals are perpendicular.

The first two of these properties are obvious from symmetry. In the kite on the right, the line of symmetry is $\overline{BD}$; this splits the kite into two triangles that are congruent by SSS (or by symmetry). Therefore, $\angle A = \angle C$, which shows the first statement. Furthermore, $\overline{BD}$ bisects $\angle B$ and $\angle D$ (by symmetry or because these are corresponding angles of congruent triangles), which shows the second statement.





To see that the diagonals are perpendicular, note that the two shaded triangles pictured on the left are congruent by SAS. Angles x and y are therefore equal, so both are 90°. Thus the diagonals are perpendicular.

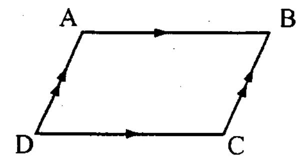
Quadrilateral Properties

We have seen that rectangles, rhombuses, and squares are also parallelograms. These figures therefore have all the properties of parallelograms. Likewise, rhombuses and squares are kites, so the properties of a kite also hold for rhombuses and squares. Below is a summary of the properties of quadrilaterals that are learned in grades K-8.

By definition, a *parallelogram* is a quadrilateral in which both pairs of opposite sides are parallel. For parallelograms,

1. opposite sides are equal,
2. opposite angles are equal,
3. the diagonals bisect each other, and
4. one pair of opposite sides are equal and parallel.

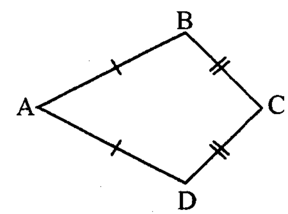
Furthermore, any quadrilateral satisfying (1), (2), (3), or (4) is a parallelogram.



By definition, a *kite* is a quadrilateral with two consecutive sides of equal length and the other two sides also of equal length. For kites,

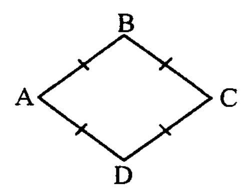
1. at least one pair of opposite angles are equal,
2. there is a diagonal that bisects a pair of opposite angles, and
3. the diagonals are perpendicular to each other.

Furthermore, any quadrilateral satisfying (2) is a kite.



By definition, a *rhombus* is a quadrilateral with all sides of equal length.

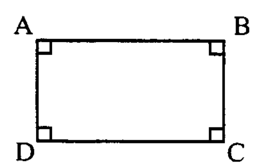
A rhombus has all the properties of a parallelogram and a kite, and the diagonals bisect the interior angles.



By definition, a *rectangle* is a quadrilateral all of whose angles are right angles.

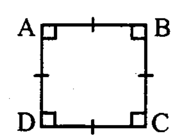
A rectangle has all of the properties of a parallelogram, and its diagonals are equal.

Furthermore, any quadrilateral whose diagonals are equal and bisect each other is a rectangle.



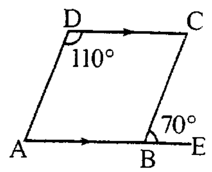
By definition, a *square* is a rectangle with all sides of equal length.

A square has the properties of a rectangle and a rhombus.

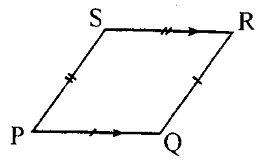


Selected Homework from Set 15

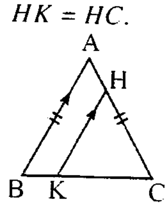
3. Give a Teacher's Solution to the following problem: In the figure, show that $ABCD$ is a parallelogram.



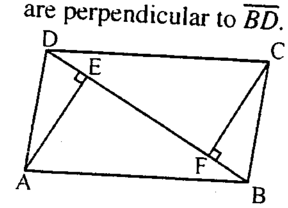
4. Give a Teacher's Solution to the following problem: In the figure, show that $PQRS$ is a rhombus.



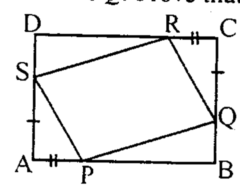
5. In the figure, $AB = AC$ and $\overline{AB} \parallel \overline{HK}$. Prove that $HK = HC$.



6. In the figure, $ABCD$ is a parallelogram, and $\overline{AE}$ and $\overline{CF}$ are perpendicular to $\overline{BD}$. Prove that $AE = CF$.



7. In the figure, $ABCD$ is a rectangle, and $AP = CR$ and $AS = CQ$. Prove that $PQRS$ is a parallelogram.



8. Give an Elementary Proof: the diagonals of a square are perpendicular. Start by drawing a picture and writing:

   Given: Square $ABCD$ with diagonals $\overline{AC}$ and $\overline{BD}$
   To Prove: $\overline{AC} \perp \overline{BD}$.

9. Show that a parallelogram with one right angle is a rectangle.