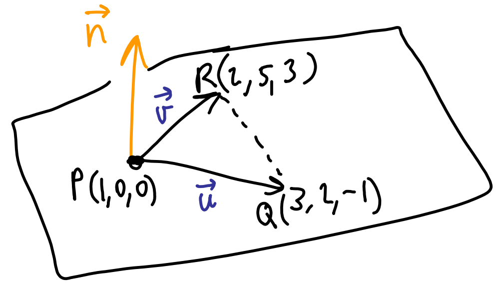
**Math 234 EXAM 1 July 13, 2007** 234.201 us07

**1.** Find the equation of the plane going through the points $(1,0,0),(3,2,-1),(2,5,3)$.
12pts

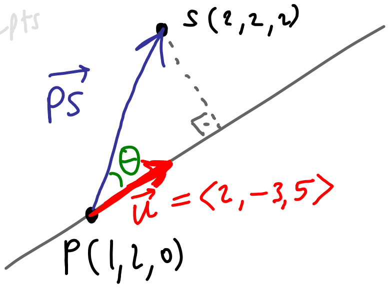



$$\vec{n} = \vec{PQ} \times \vec{PR} = \begin{vmatrix} i & j & k \\ 2 & 2 & -1 \\ 1 & 5 & 3 \end{vmatrix} = \langle 11, -7, 8 \rangle$$

Point in the plane: $(1,0,0)$

Equation: $11(x-1) - 7(y-0) + 8(z-0) = 0$

**2.** Find the distance from $(2,2,2)$ to the line given by $\mathbf{r}(t) = \langle 1 + \underline{\underline{2}}t, 2 \underline{\underline{-3}}t, \underline{\underline{5}}t \rangle$.
12pts




$$\text{distance} = |PS| \cdot \sin\theta = \frac{|PS| \cdot |u| \cdot \sin\theta}{|u|} = \frac{|PS \times u|}{|u|}$$

$$= \frac{|\langle 1,0,2 \rangle \times \langle 2,-3,5 \rangle|}{|\langle 2,-3,5 \rangle|} = \frac{\left| \begin{vmatrix} i & j & k \\ 1 & 0 & 2 \\ 2 & -3 & 5 \end{vmatrix} \right|}{\sqrt{38}}$$

$$= \frac{|\langle 6,-1,-3 \rangle|}{\sqrt{38}} = \frac{\sqrt{46}}{\sqrt{38}}$$

**3.** Given the two planes $x + 2y = 5$ and $x - y + z = 2$, find the equation of the line of intersection of the two planes.
15pts

$P_1 : \quad 1 \cdot x + 2 \cdot y + 0 \cdot z = 5 \longrightarrow \vec{n}_1 = \langle 1, 2, 0 \rangle$

$P_2 : \quad 1 \cdot x - 1 \cdot y + 1 \cdot z = 2 \longrightarrow \vec{n}_2 = \langle 1, -1, 1 \rangle$

Direction of common line lies in both planes, hence perpendicular to $n_1$ and $n_2$:

$$\vec{v} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 1 & 2 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \langle 2, -1, -3 \rangle$$

We also need a point on the line. Let's find where the line intersects xy-plane:

$z = 0$

$P_1 \cap$ xy-plane: $x + 2y = 5$
$P_2 \cap$ xy-plane: $x - y = 2$
$\Rightarrow x = 3, \ y = 1$

Common point: $(3,1,0)$

line: $\vec{r}(t) = \langle 3,1,0 \rangle + t \cdot \langle 2,-1,-3 \rangle$

$x(t) = 3 + 2t$
$y(t) = 1 - t$
$z(t) = -3t$

**4.** Given $\mathbf{r}(t) = \langle e^t cos(t), e^t sin(t), e^t \rangle$, find the velocity vector $\mathbf{v(t)}$, and compute the arc length for $-ln(4) \leq t \leq 0$. (one of suggested HW problems, 13.3.13)

12pts

$$\vec{v}(t) = \vec{r}\,'(t) = \langle e^t\cos t - e^t \sin t,\ e^t \sin t + e^t \cos t,\ e^t \rangle$$

$$|v| = \sqrt{(e^t\cos t - e^t\sin t)^2 + (e^t \sin t + e^t\cos t)^2 + (e^t)^2}$$

$$= \sqrt{e^{2t}\cos^2 t \underline{- 2e^{2t}\cos t \sin t} + e^{2t}\sin^2 t + e^{2t}\sin^2 t \underline{+ 2e^{2t}\sin t\cos t} + e^{2t}\cos^2 t + e^{2t}}$$

these terms cancel

$$= e^t\sqrt{\underbrace{\cos^2 t + \sin^2 t}_{1} + \underbrace{\sin^2 t + \cos^2 t}_{1} + 1} = e^t\sqrt{3}$$

$$\text{Arc Length} = \int_{-\ln(4)}^{0} |v|\,dt = \int_{-\ln(4)}^{0} e^t\sqrt{3}\,dt = e^t\sqrt{3}\Big|_{-\ln(4)}^{0} = \sqrt{3}\left(e^0 - e^{-\ln(4)}\right)$$

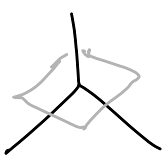
$$= \sqrt{3}\left(1 - \frac{1}{4}\right) = \frac{3\sqrt{3}}{4}$$

$$\left(e^{-\ln(4)} = \frac{1}{e^{\ln(4)}} = \frac{1}{4}\right)$$

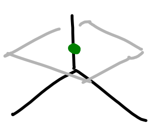
**5.** Find the cross sections and sketch the surface given by $z^2 = 1 + x^2 + 4y^2$.

10pts

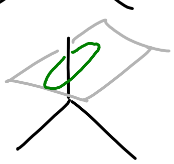
Symmetries: $x \to -x$, $y \to -y$, $z \to -z$



$z = 0$ $\quad 0 = 1 + x^2 + 4y^2 \longrightarrow$ no solution

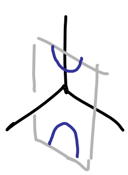


$z = 1$ $\quad 1 = 1 + x^2 + 4y^2 \longrightarrow (0,0,1)$ single point



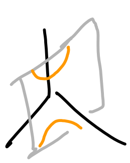
$z = 2$ $\quad 2 = 1 + x^2 + 4y^2 \longrightarrow$ ellipse, longer in x-direction

similar for $z = -2$



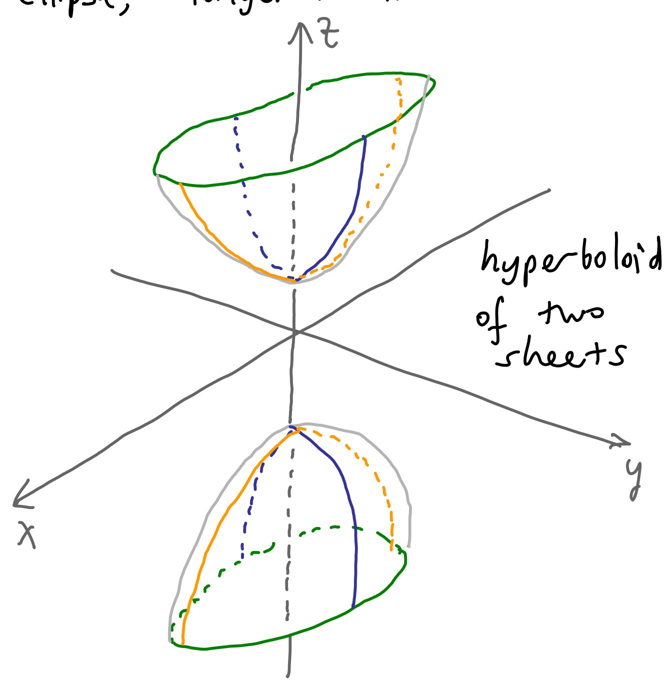
$x = 0$ $\quad z^2 = 1 + 4y^2$

hyperbola



$y = 0$ $\quad z = 1 + x^2$

hyperbola




hyperboloid of two sheets

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**6.** Given $xy + zcos(y) - 3yz = 0$, find the partial derivative $\frac{\partial z}{\partial y}$ using implicit differentiation.
15pts

Assume $x$ is constant (since it is an independent variable)

$$\frac{\partial}{\partial y}\left( xy + z\cos(y) - 3yz \right) = \frac{\partial}{\partial y}(0)$$

$$x \cdot \underbrace{\frac{\partial y}{\partial y}}_{=1} + \frac{\partial z}{\partial y}\cos y + z(-\sin y)\underbrace{\frac{\partial y}{\partial y}}_{=1} - 3\underbrace{\frac{\partial y}{\partial y}}_{=1} z - 3y\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y}\left(\cos y - 3y\right) = -x + z\sin y + 3z$$

$$\frac{\partial z}{\partial y} = \frac{-x + z\sin y + 3z}{\cos y - 3y}$$

**7.** Compute the limit $\lim\limits_{(x,y)\to(2,-4)} \frac{4x + xy + y + 4}{x^2y - xy + 4x^2 - 4x}$
12pts

$\overset{\text{plug-in}}{=} \frac{0}{0}$, hence indeterminate case.

$$= \lim_{(x,y)\to(2,-4)} \frac{\cancel{(4+y)}(x+1)}{x\cancel{(4+y)}(x-1)} = \frac{3}{2\cdot 1} = 3/2$$

**8.** Show that the limit $\lim\limits_{(x,y)\to(0,0)} \frac{x^4 - y^2}{x^4 + y^2}$ doesn't exist using the two path test.
12pts

along x-axis: $\lim\limits_{\substack{x\to 0\\ y=0}} \frac{x^4}{x^4} = 1$

along y-axis: $\lim\limits_{\substack{y\to 0\\ x=0}} \frac{-y^2}{y^2} = -1$

result depends on path, hence limit doesn't exist.

2nd way: along $y = mx$:

$$\lim_{\substack{x\to 0\\ y=mx}} \frac{x^4 - m^2x^2}{x^4 + m^2x^2} = \lim_{x\to 0} \frac{x^2 - m^2}{x^2 + m^2} = \begin{cases} -1 & m \neq 0 \\ 1 & m = 0 \end{cases}$$

3rd way: along $y = mx^2$

$$\lim_{\substack{x\to 0\\ y=mx^2}} \frac{x^4 - m^2x^4}{x^4 + m^2x^4} = \frac{1-m^2}{1+m^2}$$

depends on $m$