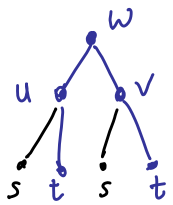
Show your work in all questions.

Math 234 EXAM 2 July 27, 2007

1. (a) State the chain rule formula for $\frac{\partial w}{\partial t}$ for $w = f(u,v)$, $u = g(s,t)$ and $v = h(s,t)$.



$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial u} \cdot \frac{\partial u}{\partial t} + \frac{\partial w}{\partial v} \cdot \frac{\partial v}{\partial t}$$

(b) Given $w = f(u,v) = 3u^5 + u \cdot ln(v)$ and $u = 2t - s$, $v = e^t$,
use chain rule to find $\frac{\partial w}{\partial t}$

$$\frac{\partial w}{\partial t} = \left(15u^4 + \ln(v)\right) \cdot 2 + \frac{u}{v} \cdot e^t$$

2. In which direction $\mathbf{u}$ is the directional derivative of $f(x,y) = xy - y^2$ at $P(2,1)$ equal to zero?

Recall: for a unit vector $\vec{u}$,

$$(D_u f)_p = \underbrace{\nabla f|_p}_{\text{gradient vector at } p} \bullet \underbrace{\vec{u}}_{\text{direction}}$$

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle = \langle y, x - 2y \rangle$$

$$\nabla f|_p = \langle 1, 2-2 \rangle = \langle 1, 0 \rangle$$

Solve $\langle 1, 0 \rangle \bullet \langle u_1, u_2 \rangle = 0$ for $\vec{u}$

$u_1 + 0 = 0 \longrightarrow u_1 = 0 \longrightarrow$ to get unit vector $u_2 = \pm 1$

answer: in the direction given by $\boxed{\vec{u} = \langle 0, 1 \rangle}$

3. Find the equation of the plane tangent to the surface $z = x^2 - 3y^2$ at $P(1, 0, 1)$.

$f(x,y,z) = x^2 - 3y^2 - z$

$f(x,y,z) = 0$ defines the given surface. ∇f gives the normal.

$\nabla f = \langle 2x, -6y, -1 \rangle$

$\vec{n} = \nabla f|_P = \langle 2, 0, -1 \rangle$

eqn. of tangent line: $2(x-1) + 0(y-0) - (z-1) = 0$

$$\boxed{2x - z = 1}$$

4. Find the absolute maximum and absolute minimum of the function

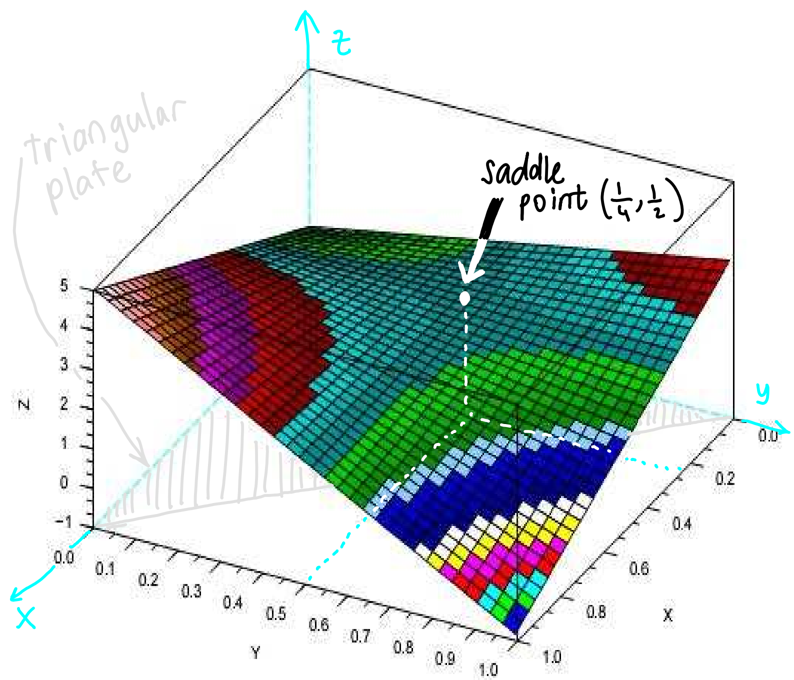
$$f(x, y) = 4x - 8xy + 2y + 1$$

on the triangular plate bounded by the lines $x = 0$, $y = 0$ and $x + y = 1$ in the first quadrant.

$f_x = 4 - 8y \qquad f_y = -8x + 2$

critical point:
$f_x = f_y = 0 \longrightarrow (x,y) = \left(\frac{1}{4}, \frac{1}{2}\right)$

$f\left(\frac{1}{4}, \frac{1}{2}\right) = 2$



Also need to check f on boundary

When $x = 0$: $f(0,y) = 2y + 1$
has no critical points, only boundary:
$f(0,0) = \underline{1}$, $\quad f(0,1) = 3$

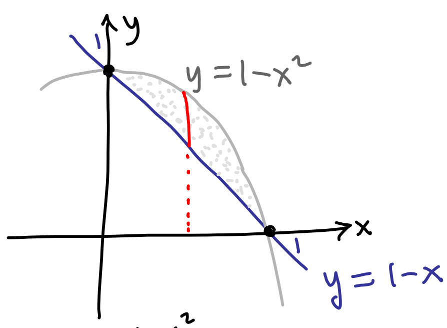
When $y = 0$: $f(x,0) = 4x + 1$, $\quad f(0,0) = 1$, $\quad f(1,0) = \underline{5}$

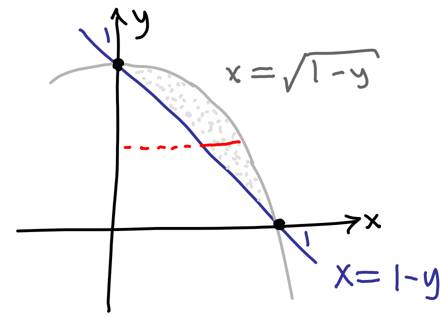
When $x + y = 1$, $f(x, 1-x) = 4x - 8x(1-x) + 2(1-x) + 1$
$= 8x^2 - 6x + 3 = g(x) \longrightarrow g'(x) = 16x - 6$
$x = \frac{3}{8}, y = \frac{5}{8}$

$f\left(\frac{3}{8}, \frac{5}{8}\right) = 15/8$

Hence absolute minimum is $\underline{1}$, absolute maximum is $\underline{5}$.

5. Express the double integral $\iint_R x^2 + x\cos(y)dA$ as an iterated integral in two different orders of integration on the region R which lies above $x + y = 1$ and below $y = 1 - x^2$. (Do not evaluate)

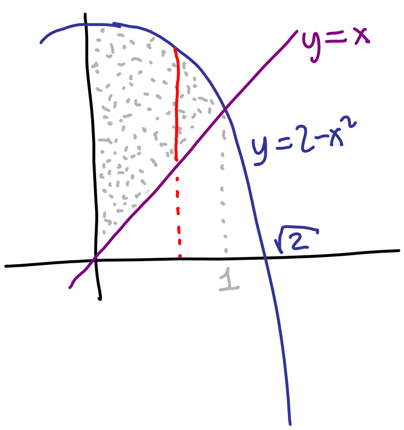




$$\int_{x=0}^{1}\int_{y=1-x}^{1-x^2} x^2 + x\cos(y)\,dy\,dx$$

$$\int_{y=0}^{1}\int_{x=1-y}^{\sqrt{1-y}} x^2 + x\cos(y)\,dx\,dy$$

6. Find the center of mass for the thin plate of constant density bounded by $x = 0$, $y = x$ and the parabola $y = 2 - x^2$ in the first quadrant. (Hint: first sketch region and find mass and first moments)



$$\text{Mass} = \iint_R 1\cdot dA = \int_{x=0}^{1}\int_{y=x}^{2-x^2} dy\,dx = \int_0^1 2 - x^2 - x\,dx$$

$$= 2x - \frac{x^3}{3} - \frac{x^2}{2}\Big|_0^1 = 2 - \frac{1}{3} - \frac{1}{2} = \frac{7}{6}$$

$$M_y = \iint_R x\,dA = \int_0^1\int_{y=x}^{2-x^2} x\,dy\,dx = \int_0^1 x\cdot\int_{y=x}^{2-x^2} dy\,dx$$

$$= \int_0^1 x\cdot(2 - x^2 - x)\,dx = x^2 - \frac{x^4}{4} - \frac{x^3}{3}\Big|_0^1 = \frac{5}{12}$$

$$M_x = \iint_R y\,dA = \int_0^1\int_{y=x}^{2-x^2} y\,dy\,dx = \int_0^1 \left(\frac{y^2}{2}\right)\Big|_x^{2-x^2} dx$$

$$= \frac{1}{2}\int_0^1 (2-x^2)^2 - x^2\,dx = \frac{1}{2}\int_0^1 4 - 4x^2 + x^4 - x^2\,dx$$

$$= \frac{1}{2}\left(4x - 5\frac{x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1 = \frac{1}{2}\cdot\frac{38}{15} = \frac{19}{15}$$

Center of mass: $(\bar{x}, \bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right) = \left(\frac{5/12}{7/6}, \frac{19/15}{7/6}\right) = \left(\frac{5}{14}, \frac{38}{35}\right)$

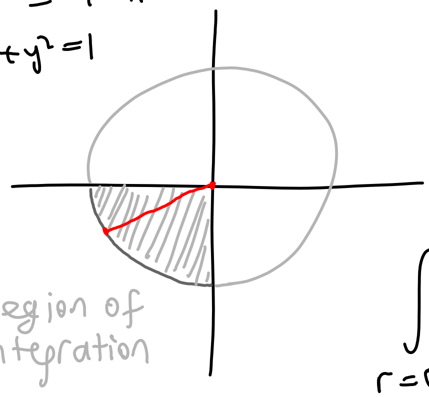
7. Convert the following integral into an equivalent polar integral, then evaluate the polar integral.

$$\int_{x=-1}^{0}\int_{y=-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\, dy\,dx$$

$y = -\sqrt{1-x^2}$

$y^2 = 1-x^2$

$x^2+y^2=1$



region of integration

$x = r\cos\theta$

$y = r\sin\theta$

$x^2+y^2 = r^2(\cos^2\theta+\sin^2\theta) = r^2$

$dy\,dx = r\,dr\,d\theta$

$$\int_{r=0}^{1}\int_{\theta=\pi}^{3\pi/2} \frac{2}{1+r}\cdot r\,d\theta\,dr = \int_{r=0}^{1} \frac{\pi}{\cancel{2}}\cdot\frac{\cancel{2}r}{1+r}\,dr$$

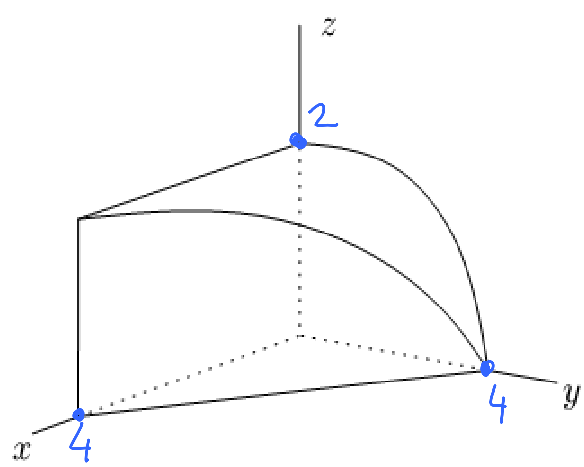
$$= \pi\int_{r=0}^{1} \frac{r+1-1}{1+r}\,dr = \pi\int_0^1 1\,dr - \pi\int_0^1 \frac{1}{1+r}\,dr$$

$$= \pi r\Big|_0^1 - \pi\ln(1+r)\Big|_0^1 = \pi - \pi\big(\ln(2) - \underbrace{\ln(1)}_{=0}\big)$$

$$= \pi\,(1-\ln(2))$$

8. Express the volume of the region in the first octant bounded by the coordinate planes, the plane $x+y=4$ and the cylinder $y^2+4z^2=16$ as a triple integral. Do not evaluate.

$$\int_{x=0}^{4}\int_{y=0}^{4-x}\int_{z=0}^{\frac{\sqrt{16-y^2}}{2}} dz\,dy\,dx$$



equivalently:

$$\int_{y=0}^{4}\int_{z=0}^{\frac{\sqrt{16-y^2}}{2}}\int_{x=0}^{4-y} dx\,dz\,dy$$

not all of them are easy to write:

$$\int_{x=0}^{4}\int_{z=0}^{2}\int_{y=0}^{???} dy\,dz\,dx$$