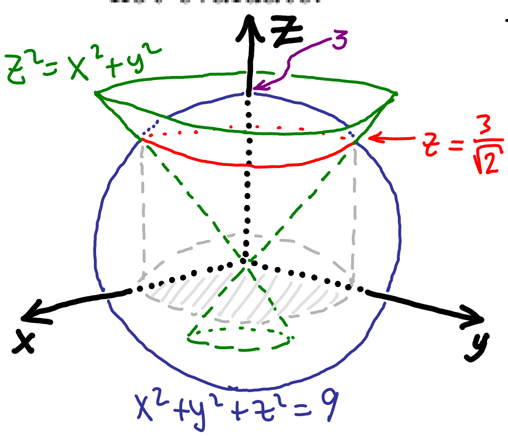
Math 234 S A M P L E E X A M 3 *Solutions* Section 201 summer 07

Show your work in all questions.

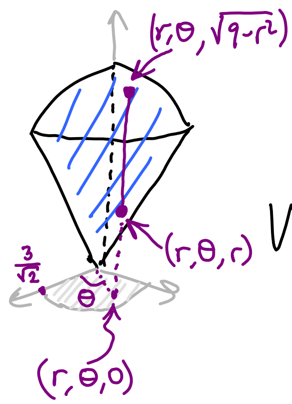
1. Let D be the region in the first octant that is bounded below by the cone $z^2 = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 9$.

(a) Express the volume of D as a triple integral in **cylindrical** coordinates. Do not evaluate.



intersection:
$$\left.\begin{aligned} z^2 &= x^2+y^2 \\ x^2+y^2+z^2 &= 9 \end{aligned}\right\} \quad \begin{aligned} 2z^2 &= 9 \\ z &= \pm 3/\sqrt{2} \end{aligned}$$

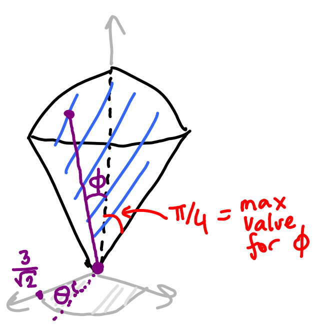
The solid shape between the two surfaces in the first octant looks as follows:



It projects to a quarter of a disk in the xy-plane.

$$\text{Volume} = \int_{\theta=0}^{\pi/2} \int_{r=0}^{3/\sqrt{2}} \int_{z=r}^{\sqrt{9-r^2}} \underbrace{r\,dz\,dr\,d\theta}_{dV}$$

(b) Express the volume of D as a triple integral in **spherical** coordinates. Do not evaluate.



Cross sections in spherical coordinates are arcs along rays from the origin

$$\int_{\phi=0}^{\pi/4} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{3} \underbrace{\rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi}_{dV}$$

2. Integrate the function $f(x, y, z) = 1+x-z+\sqrt{y}$ over the path $r(t) = \langle t, t^2, 1-t\rangle$ from $(1, 1, 0)$ to $(3, 9, -2)$.

$$r'(t) = \langle 1, 2t, -1\rangle$$

$$\int_C f(x,y,z)\,ds = \int_{t=1}^{3} (x^2y - xz)\cdot |r'(t)|\,dt$$

in actual exam the integral will be easier.

$$= \int_1^3 \left(1+t-(1-t)+\underbrace{\sqrt{t^2}}_{\substack{\| \\ t \\ \text{since } t>0}}\right)\cdot\sqrt{1^2+(2t)^2+(-1)^2}\,dt = \int_1^3 (t+t+t)\cdot\sqrt{2+4t^2}\,dt$$

$$= \int_{t=1}^{3} 3t\cdot\sqrt{2+4t^2}\,dt = \int_6^{38} \frac{3}{8}\sqrt{u}\,du = \frac{3}{8}\,\frac{u^{3/2}}{3/2}\Bigg|_6^{38} = \frac{3}{4}u^{3/2}\Bigg|_6^{38} = \frac{3}{4}\left(38^{3/2} - 6^{3/2}\right)$$

$u = 2+4t^2$
$du = 8t\,dt$

3. Find the work done by the vector field $F = \langle y-x, z-y, x^2+y^2\rangle$ over the space curve $r(t) = \langle cos(t), sin(t), t\rangle$ for $0 \le t \le \pi$.

$$r'(t) = \langle -\sin(t), \cos(t), 1\rangle$$

$$\text{Work} = \int_C F\cdot T\,ds = \int_{t=0}^{\pi} F\cdot \frac{dr}{dt}\,dt = \int_0^{\pi} \langle y-x, z-y, x^2+y^2\rangle \cdot \langle -\sin(t), \cos(t), 1\rangle\,dt$$

$$= \int_0^{\pi} (\sin t - \cos t)(-\sin t) + (t - \sin t)\cos t + (\cos^2 t + \sin^2 t)\cdot 1 \; dt$$

$$= \int_0^{\pi} -\sin^2 t + \cos t \sin t + t\cos t - \sin t \cos t + 1 \; dt = \int_0^{\pi} \underbrace{1-\sin^2 t}_{\cos^2 t = \frac{1+\cos 2t}{2}}\,dt + \boxed{\int_0^{\pi} \underbrace{t\cos t\,dt}_{\text{need integration by parts}}}$$

$u = t \quad \cos t\,dt = dv$

$du = 1\cdot dt \quad \sin t = v$

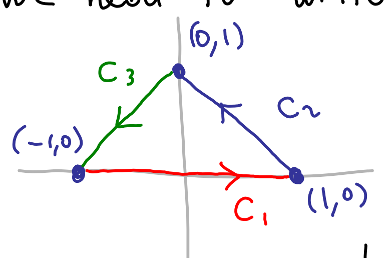
$$= \frac{1}{2}\left(t + \frac{1}{2}\sin 2t\right)\Big|_0^{\pi} + \boxed{t\cdot \sin t\Big|_0^{\pi} - \int_0^{\pi} \sin t\,dt}$$

$$= \frac{1}{2}(\pi - 0) + 0 - (-\cos t)\Big|_0^{\pi} = -1-1 = -2$$

4. Compute the flux of the vector field $F = \langle \underbrace{x+y}_{M}, \underbrace{-(x^2+y^2)}_{N}\rangle$ across the triangle with vertices $(1,0)$, $(0,1)$, $(-1,0)$.

$$\text{Flux} = \int_C F\cdot n\,ds = \int_C M\,dy - N\,dx$$

We need to write $C = C_1 \cup C_2 \cup C_3$ and parametrize each curve:



C_1: $r(t) = \langle t, 0\rangle \qquad -1 \le t \le 1$

C_2: $r(t) = \langle 1-t, t\rangle \qquad 0 \le t \le 1$

C_3: $r(t) = \langle -t, 1-t\rangle \qquad 0 \le t \le 1$

Flux across C_1: $\displaystyle\int_{t=-1}^{1} M\,dy - N\,dx = \int_{-1}^{1} (t+0)\cdot 0\,dt - (-t^2)\,dt = \frac{t^3}{3}\Big|_{-1}^{1} = \frac{1}{3} - \frac{-1}{3} = \frac{2}{3}$

Flux across C_2: $\displaystyle\int_{t=0}^{1} (1-t+t)\,dt - \left(-\left((1-t)^2+t^2\right)\right)(-dt) = \int_0^1 1 - (1-2t+2t^2)\,dt = t^2 - \frac{2t^3}{3}\Big|_0^1 = \frac{1}{3}$

Flux across C_3: $\displaystyle\int_{t=0}^{1} (-t+1-t)(-dt) - \left(-\left((-t)^2+(1-t)^2\right)\right)(-dt)$

$$= \int_0^1 2t - 1 - (2t^2-2t+1)\,dt = \int_0^1 4t - 2t^2 - 2\,dt = 2t^2 - \frac{2t^3}{3} - 2t\Big|_0^1$$

$$= 2 - \frac{2}{3} - 2 = -2/3$$

Total flux $= 2/3 + 1/3 - 2/3 = 1/3$

Alternatively: Use Green's theorem: $\displaystyle\int_{y=0}^{1}\int_{x=-1+y}^{1-y} M_x + N_y\,dx\,dy = \int_{y=0}^{1}\int_{x=-1+y}^{1-y} 1-2y\,dx\,dy = 1/3$

$$= \int_{y=0}^{1} (1-2y)(2-2y)\,dy = \int_{y=0}^{1} 2 - 6y + 4y^2\,dy = 2y - \frac{6y^2}{2} + \frac{4y^3}{3}\Big|_0^1 = 1/3$$

5. Apply the component test to show $F = \langle \underbrace{ysin(z)}^{M}, \underbrace{xsin(z)}^{N}, \underbrace{xycos(z)}^{P} \rangle$ is a conservative vector field. Find its potential function $f(x, y, z)$.

$$M_y = \sin z = N_x \checkmark$$
$$M_z = y\cos(z) = P_x \checkmark$$
$$N_z = x\cos(z) = P_y \checkmark$$

hence $F = \nabla f$ for some $f(x,y,z)$

data: $\frac{\partial f}{\partial x} = y\sin(z) \longrightarrow f(x,y,z) = xy\sin(z) + g(y,z)$

$\downarrow$

$\frac{\partial f}{\partial y} = x\sin(z) \quad = \quad f_y = x\sin z + g_y(y,z) \longrightarrow g_y = 0$

$\frac{\partial f}{\partial z} = xy\cos(z)$

$f = xy\sin(z) + h(z)$

$\downarrow$

$f_z = xy\cos(z) + h_z$

we see $h_z = 0$, hence $f(x,y,z) = xy\sin(z) + C$

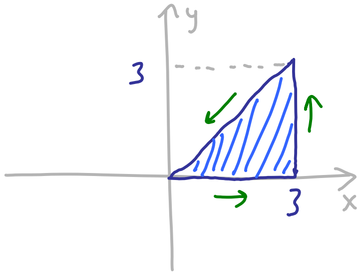
6. Evaluate the work integral $\int_C F \cdot dr$ for the force field $F = \langle x^3, e^y \rangle$ on the circle $r(t) = \langle 2cos(t), 2sin(t) \rangle$ between $t = 0$ and $t = \frac{\pi}{2}$. Hint: This problem has a short solution.

Note that F is conservative: $F = \nabla f$ with $f(x,y) = \frac{x^4}{4} + e^y$

hence

$$\int_C F \cdot dr = f(B) - f(A) = f(r(\pi/2)) - f(r(0))$$
$$= f(0,2) - f(2,0) = e^2 - \left(\frac{2^4}{4} + 1\right)$$
$$= e^2 - 5$$

7. Using Green's theorem, find the outward flux for $F = \langle y^2 - x^2, x^2 + y^2 \rangle$ on the triangle bounded by $y = 0, x = 3, y = x$.

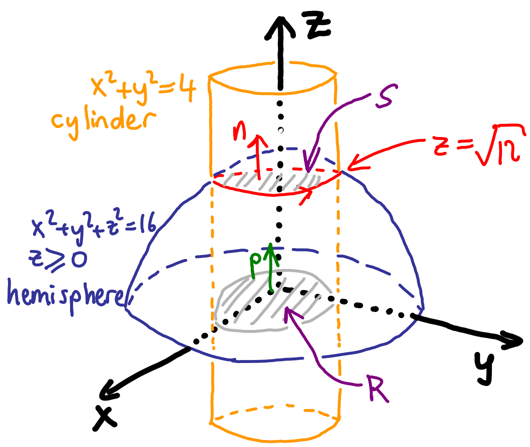


$$\oint_C F \cdot n \, dr = \iint_R M_x + N_y \; dy\,dx$$

$$= \int_{x=0}^{3} \int_{y=0}^{x} -2x + 2y \; dy\,dx = \int_{x=0}^{3} \left(-2xy + y^2\right)\Big|_0^x \, dx$$

$$= \int_0^3 \underbrace{-2x^2 + x^2}_{-x^2} \, dx = -\frac{x^3}{3}\Big|_0^3 = -9$$

8. Use the surface integral in Stokes' theorem to calculate the circulation of the vector field $F = \langle x^2y^3, 1, z \rangle$ on the intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16, z \geq 0$ counterclockwise when viewed from above.



The two surfaces intersect along the circle
C: $x^2+y^2=4$, $z=\sqrt{12}$
A surface that bounds this circle is the disk $x^2+y^2 \leq 4$, $z=2$

Hence $\oint_C F \cdot dr = \iint_S (\nabla \times F) \cdot n \, d\sigma$

$$= \iint_R (\nabla \times F) \cdot n \cdot \frac{|\nabla g|}{|\nabla g \cdot p|} \, dx\,dy$$

Note that $\vec{n}$ = normal to S $= \vec{k}$, $\quad \vec{p}$ = normal to projection R of S $= \vec{k}$

$g(x,y,z) = z - \sqrt{12}$ (since $z=2$ defines the plane S lies in)

$\nabla g = \langle 0, 0, 1 \rangle$, $\quad |\nabla g| = 1$, $\quad \nabla g \cdot p = 1$

$$(\nabla \times F) \cdot \vec{n} = (\nabla \times F) \cdot \vec{k} = k\text{-component of } \underbrace{\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix}}_{i(0-0) - j(0-0) + k(0 - 3x^2y^2)}$$

$$= -3x^2y^2$$

$$\text{Flux} = \iint_R -3x^2y^2 \, dx\,dy = \int_{\theta=0}^{2\pi} \int_{r=0}^{2} -3(r\cos\theta)^2 (r\sin\theta)^2 \, r\,dr\,d\theta$$

$$= \int_0^{2\pi} -3 \cos^2\theta \sin^2\theta \cdot \frac{r^6}{6}\Big|_0^2 \, d\theta = \int_0^{2\pi} -\frac{3}{4} \cdot \underbrace{\sin^2 2\theta}_{\frac{1-\cos 4\theta}{2}} \cdot \frac{64}{6} \, d\theta = -8\left(\theta - \frac{\sin 4\theta}{4}\right)\Big|_0^{2\pi}$$

$$= -16\pi$$