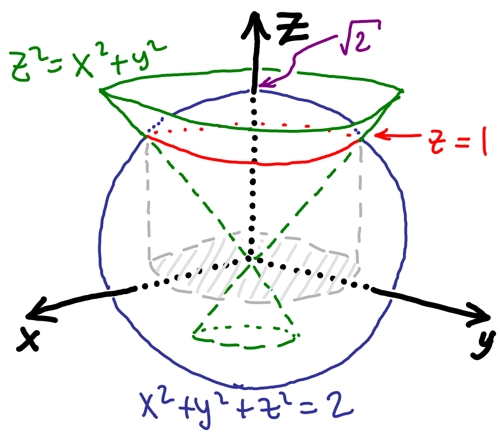
Math 234 Exam 3 Solutions

Section 201
Summer 07

(1) Let D be the region in the upper half space $z \geq 0$ that is bounded below by the cone $z^2 = x^2+y^2$ and above by the sphere $x^2+y^2+z^2=2$.

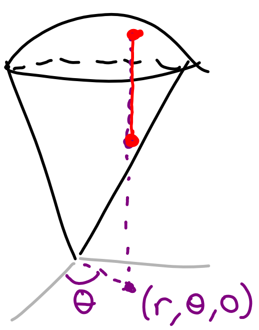
(a) Sketch the solid and find where the sphere intersects the cone.



$$x^2+y^2+z^2=2$$
$$z^2=x^2+y^2$$

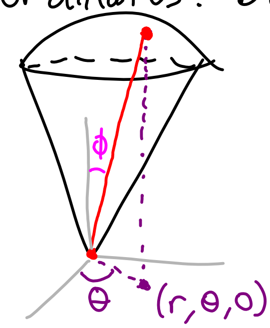
$$2z^2=2$$
$$z^2=1$$
$$z=\pm 1$$

(b) Express the volume of D as a triple integral in cylindrical coordinates. Do not evaluate.



$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1}\int_{z=r}^{\sqrt{2-r^2}} r\,dz\,dr\,d\theta$$

(c) Express the volume of D as a triple integral in spherical coordinates. Do not evaluate.



radius of sphere

$$\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/4}\int_{\rho=0}^{\sqrt{2}} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

② Integrate the function $f(x,y,z) = x - y + z - 2$ over the straight line segment from $(0,1,0)$ to $(2,0,1)$.

$$r(t) = \langle 0,1,0\rangle + t\cdot\left(\langle 2,0,1\rangle - \langle 0,1,0\rangle\right)$$

$$= \langle 2t,\ 1-t,\ t\rangle$$

$$r'(t) = \langle 2, -1, 1\rangle, \qquad |r'(t)| = \sqrt{6}$$

$$\int_C f(x,y,z)\,ds = \int_{t=0}^{1} f(r(t))\cdot \underbrace{|r'(t)|\,dt}_{ds}$$

$$= \int_0^1 (2t - (1-t) + t - 2)\cdot\sqrt{6}\,dt = \sqrt{6}\int_0^1 4t-3\,dt$$

$$= \sqrt{6}\left(2t^2 - 3t\right)\Big|_0^1 = -\sqrt{6}$$

③ Find the work done by the vector field $F = \langle 3x^2-3x, 3z, 1\rangle$ over the space curve $r(t) = \langle t, t^2, t^4\rangle$ for $0 \le t \le 1$.

$r'(t) = \langle 1, 2t, 4t^3\rangle$ $\quad F \neq \nabla f$ since $N_z \neq P_y$

$$\text{Work} = \int_C F\cdot T\,ds = \int_{t=0}^{1} F\cdot r'(t)\,dt = \int_0^1 \langle 3t^2-3t, 3t^4, 1\rangle\cdot\langle 1, 2t, 4t^3\rangle dt$$

$$= \int_0^1 3t^2 - 3t + 6t^5 + 4t^3\,dt = t^3 - \frac{3t^2}{2} + t^6 + t^4\Big|_0^1 = 3 - \frac{3}{2} = \frac{3}{2}$$

④ Compute the flux $\int_C F\cdot n\,ds$ of the vector field $F = \langle -y^2, x^2\rangle$ across the triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$.

$$\text{Flux} = \int_C F\cdot n\,ds = \int_C N\,dx - M\,dy \overset{\text{Green's thm}}{=} \iint_R M_x + N_y\,dx\,dy = \iint_R 0 + 0\,dx\,dy$$

$$= 0$$

⑤ Apply component test to show

$$F = \langle 2xe^{xy} + x^2 y e^{xy},\quad x^3 e^{xy} + 2y,\quad e^z \rangle$$

is a conservative vector field. Find its potential function $f(x,y,z)$.

$$M_y = 2x^2 e^{xy} + x^2 e^{xy} + x^3 y e^{xy}$$

same

$$N_x = 3x^2 e^{xy} + x^3 y e^{xy}$$

$$M_z = P_x = 0, \qquad N_z = P_y = 0$$

Potential: we're looking for $f(x,y,z)$ with $\nabla f = F$

easiest start: $f_y = N$

$$f(x,y,z) = \int x^3 e^{xy} + 2y \, dy$$

$$= x^2 e^{xy} + y^2 + g(x,z)$$

$$f_x = M = 2xe^{xy} + x^2 y e^{xy}$$

$$f_x = 2xe^{xy} + x^2 y e^{xy} + g_x(x,z) \longrightarrow g_x(x,z) = 0$$

$$g(x,z) = h(z)$$

$$f_z = P = e^z$$

$$f_z = h_z \longrightarrow h(z) = e^z + c$$

hence $f(x,y,z) = x^2 e^{xy} + y^2 + e^z + c$ ← constant real number

⑥ Evaluate the work integral $\int_C F \cdot dr$ for the force field $F = \langle 4\sin(x), 12y^5 \rangle$ on the curve $r(t) = \langle t, 1-t^2 \rangle$ between $t=0$ and $t=\pi$.

Note that $F = \nabla f$ with $f(x,y) = -4\cos(x) + 2y^6$

hence

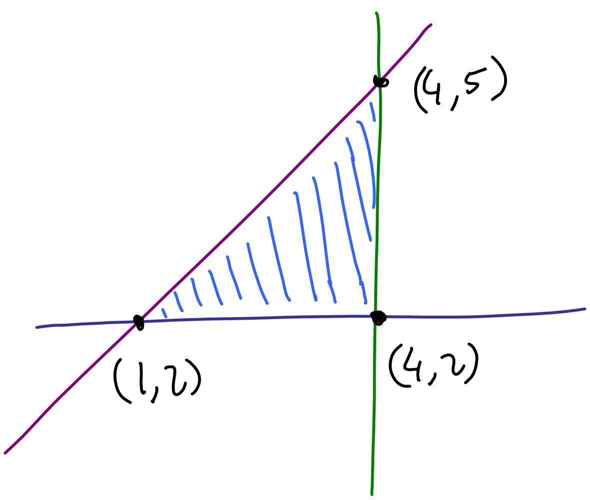
$$\int_C F \cdot dr = f(r(\pi)) - f(r(0))$$

$$= \left(-4\cdot(-1) + 2(1-\pi)^6\right) - \left(-4\cdot 1 + 2\cdot 1\right)$$

$$= 6 + 2(1-\pi)^6$$

⑦ Using Green's theorem, find the outward flux for $F = \langle x^2 - 5y, x^2 - y^2 \rangle$ on the triangle bounded by

$y=2$, $x=4$, $y=x+1$.



$$\text{Flux} = \oint F \cdot n \, dr = \iint_R M_x + N_y \, dy \, dx$$

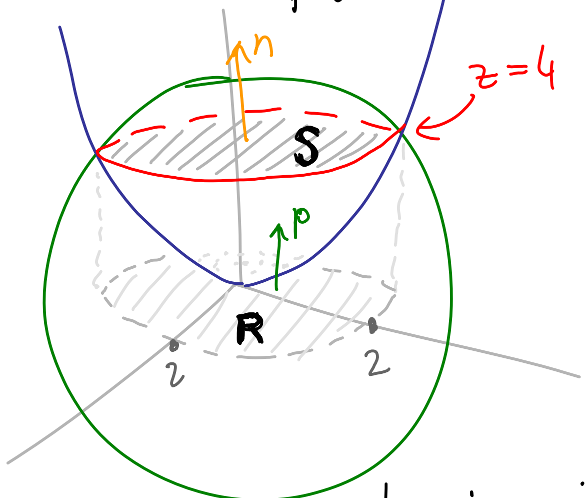
$$= \int_{x=1}^{4} \int_{y=2}^{x+1} 2x - 2y \, dy \, dx$$

$$= \int_{x=1}^{4} \left(2xy - y^2\right)\Big|_2^{x+1} dx$$

$$= \int_1^4 x^2 - 4x + 3 \, dx = \frac{x^3}{3} - 2x^2 + 3x \Big|_1^4 = \left(\frac{64}{3} - 32 + 12\right) - \left(\frac{1}{3} - 2 + 3\right)$$

$$= 0$$

⑧ Use the surface integral in Stokes' theorem to calculate the circulation of the vector field $F = \langle x^2 - y, 3x, 4y \rangle$ on the intersection of the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 20$ oriented counterclockwise when viewed from above.



need to compute the surface integral:

$\vec{n} = \vec{k}$

$\vec{p} = \vec{k}$

$d\sigma = dx\,dy$

$$\text{curl} = \nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 - y & 3x & 4y \end{vmatrix} = \langle 4, 0, 4 \rangle$$

$$\text{circulation} = \int F \cdot dr \overset{\text{Stokes}}{=\!=} \iint_R 4 \, dx\,dy = 4 \cdot \text{area}(R)$$

$$= 4 \cdot \pi \, 2^2 = 16\pi$$