Math 234 FINAL EXAM August 15, 2007

Section 201 US 2007

1. [18 points] Given the points $A(1,2,1)$, $B(2,3,-1)$ and $C(3,3,3)$,

(a) Find the vectors **AB** and **AC**.

$$\vec{AB} = \langle 2-1, 3-2, -1-1 \rangle = \langle 1, 1, -2 \rangle$$
$$\vec{AC} = \langle 3-1, 3-2, 3-1 \rangle = \langle 2, 1, 2 \rangle$$

(b) Find $cos(\theta)$ for the angle θ between **AB** and **AC**

$$AB \cdot AC = |AB| \cdot |AC| \cdot \cos(\theta)$$
$$1\cdot 2 + 1\cdot 1 + (-2)\cdot 2 = \sqrt{1^2+1^2+(-2)^2} \cdot \sqrt{2^2+1^2+2^2} \cdot \cos\theta$$
$$\cos\theta = \frac{-1}{\sqrt{6}\cdot\sqrt{9}} = \frac{-1}{3\sqrt{6}}$$

(c) Find the area of the triangle with vertices at A, B and C.

$$\text{area} = \frac{1}{2}|AB \times AC| = \frac{1}{2}\cdot\left| \begin{vmatrix} i & j & k \\ 1 & 1 & -2 \\ 2 & 1 & 2 \end{vmatrix} \right| = \frac{1}{2}|\langle 4, -6, -1 \rangle| = \frac{\sqrt{53}}{2}$$

2. [16 points] Find the parametric equation for the line going through $P(1,1,2)$ and perpendicular to the plane $\underbrace{3x - 2y + z = 5.}_{n = \langle 3, -2, 1 \rangle}$

$n=\langle 3,-2,1\rangle$ = direction vector for line

$$r(t) = P + t\cdot\vec{v} = \langle 1+3t, 1-2t, 2+t \rangle$$

3. [16 points] A particle is moving at velocity given by $\mathbf{v}(t) = \langle 2t, 6t^2 - 4t \rangle$. At time $t = 0$ the particle is at $(2, 1)$.

(a) Write a formula for the position of the particle at time t.

$$r(t) = P_0 + \int_0^t v(s)\,ds = \langle 2 + t^2, 1 + 2t^3 - 2t^2 \rangle$$

(b) Write an integral for the arc length of $\mathbf{r}(t)$ for $0 \le t \le 2$. (Do not evaluate)

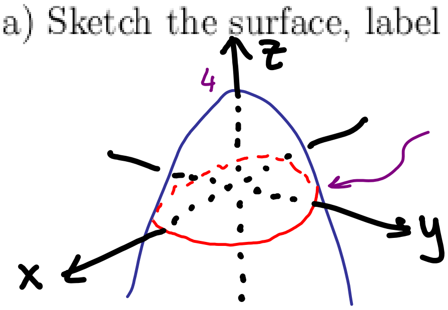
$$\text{arc length} = \int_0^2 |v(t)|\,dt = \int_0^2 \sqrt{4t^2 + (6t^2 - 4t)^2}\,dt$$

(c) Find the unit tangent vector to $\mathbf{r}(t)$ at $t = 2$.

$$u = \frac{v(2)}{|v(2)|} = \frac{1}{\sqrt{4^2 + 16^2}} \langle 4, 16 \rangle = \frac{1}{\sqrt{272}} \cdot \langle 4, 16 \rangle$$

4. [18 points] Given the surface $z = 4 - x^2 - y^2$,

(a) Sketch the surface, label important points and intersection with xy-plane.



(b) Find the equation of the tangent plane to the surface at $(2, 1, -1)$.

$$g(x, y, z) = 4 - x^2 - y^2 - z$$

$$\nabla g = \langle -2x, -2y, -1 \rangle$$

$$\text{normal} = \langle -4, -2, -1 \rangle$$

Tangent plane: $-4(x-2) - 2(y-1) - (z+1) = 0$

$$-4x - 2y - z + 9 = 0$$

5. [16 points] Given $w = u^2 - 3uv$ and $u = e^x - \cos(y)$, $v = xy - y^3$, apply chain rule to find $\dfrac{\partial w}{\partial x}$. Express your final answer in terms of x and y only.

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\cdot\frac{\partial v}{\partial x} = (2u-3v)\cdot e^x + (-3u)\cdot y$$

$$= (2e^x - 2\cos(y) - 3xy + 3y^3)e^x + (-3e^x + 3\cos y)\cdot y$$

6. [18 points] Given $f(x,y) = 4x - x^2 - y^2$,

(a) Find all critical points and identify if they are local minimum, maximum or saddle points.

$f_x = 4 - 2x$ crit pts: $y = 0$, $x = 2 \rightarrow (2,0)$

$f_y = -2y$

$f(2,0) = \underline{\underline{4}}$

$f_{xx} = -2$

$f_{yy} = -2$

$f_{xy} = 0$

$f_{xx}\cdot f_{yy} - f_{xy}^2 > 0$

and

$f_{xx} < 0$

hence it's a local max

(b) Express f in polar coordinates (r,θ), and find absolute maximum and absolute minimum of f over the closed disk $x^2 + y^2 \leq 9$.

$$f(r,\theta) = 4r\cos\theta - r^2$$

on the boundary $f(3,\theta) = 12\cos\theta - 9$

max. value on boundary: $\underline{\underline{3}}$

min. value on boundary: $\underline{\underline{-21}}$

Abs max: $\underline{\underline{4}}$ at $(x,y) = (2,0)$

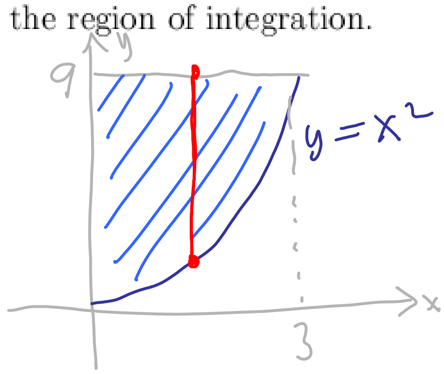
Abs min: $\underline{\underline{-21}}$ at $(x,y) = (-3,0)$

$\updownarrow$

$(r,\theta) = (3,\pi)$

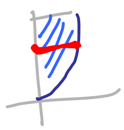
7. [16 points] Given the double integral $\int_0^3 \int_{x^2}^9 x^3 \cos(y^3)\,dy\,dx$,

(a) Sketch the region of integration.



$\longleftrightarrow \quad x=\sqrt{y}$

(b) Rewrite the integral reversing the order of integration. (Do not integrate)



$$\int_{y=0}^{9} \int_{x=0}^{\sqrt{y}} x^3 \cos(y^3)\,dx\,dy$$

8. [16 points] For the vector field $\mathbf{F} = \langle 2xy + y\sin(z),\ x^2 + x\sin(z),\ xy\cos(z)\rangle$,

(a) ($\mathbf{F}$ is conservative) Find a potential function $f(x, y, z)$ for $\mathbf{F}$.

$f_x = 2xy + y\sin(z) \longrightarrow f(x,y,z) = x^2y + xy\sin z + g(y,z)$

$N = x^2 + x\sin(z) = f_y = x^2 + x\sin(z) + g_y \longrightarrow g_y = 0,$

$P = xy\cos(z) = f_z = xy\cos(z) + h_z \qquad g(y,z) = h(z)$

$\longrightarrow h_z = 0, \quad h = \text{constant}$

$f(x,y,z) = x^2y + xy\sin(z) + C$

Verify: $\nabla f = \langle 2xy + y\sin(z),\ x^2 + x\sin(z),\ xy\cos(z)\rangle$

(b) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$ on the line segment joining $(0,0,0)$ to $(1,2,\pi)$.

$$\int_C F\cdot dr = f(1,2,\pi) - f(0,0,0)$$

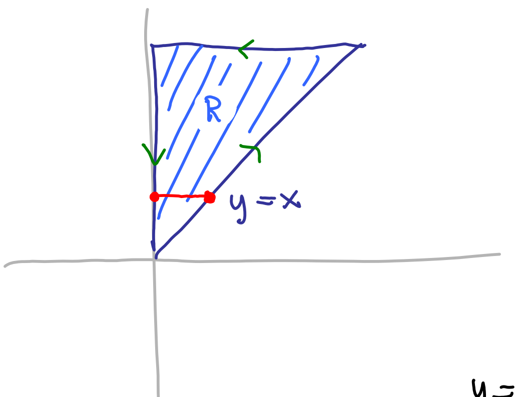
$$= 2 + 2\sin(\pi) - (0 + 0\cdot\sin(0)) = 2$$

9. [16 points] (a) State Green's theorem for integrals of the form $\int_C M dx + N dy$ where C is a closed curve that bounds a region R.

$$\oint_C M\,dx + N\,dy = \iint_R N_x - M_y \, dx\,dy$$

Where R is the region bounded by the closed curve C

(b) Evaluate the integral $\int_C (2xy + y^2)dx + (x^2 - y^4)dy$ where C is the triangle with vertices $(0, 0)$, $(1, 1)$, $(0, 1)$ traversed counterclockwise.



$$M = 2xy + y^2 \longrightarrow M_y = 2x + 2y$$

$$N = x^2 - y^4 \longrightarrow N_x = 2x$$

$$\int_{y=0}^{1} \int_{x=0}^{y} 2x - (2x + 2y)\, dx\, dy$$

$$= \int_{y=0}^{1} \int_{x=0}^{y} -2y\, dx\, dy = \int_0^1 -2yx \Big|_0^y dy = \int_0^1 -2y^2\, dy = -2\frac{y^3}{3}\Big|_0^1 = -\frac{2}{3}$$

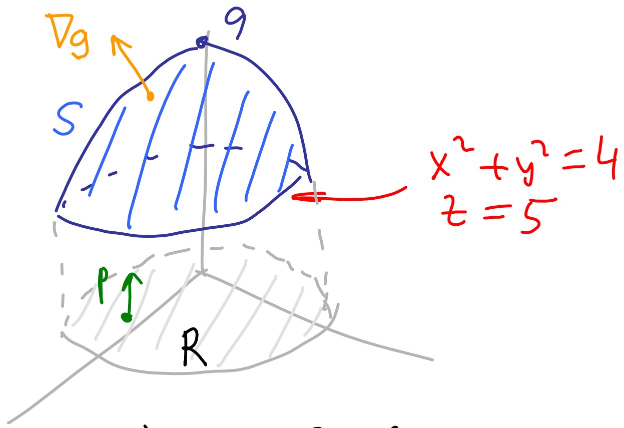
10. [16 points] Apply the two path test to show $\lim_{(x,y)\to(0,0)} \frac{x^2 + y}{x^2 - y}$ doesn't exist.

let $y = kx^2$ $\quad \lim_{x\to 0} \frac{x^2 + k x^2}{x^2 - k x^2} = \frac{1+k}{1-k}$

depends on k

hence limit doesn't exist

11. [18 points] Find the surface area of the portion of the paraboloid $z = 9 - x^2 - y^2$ that lies above the plane $z = 5$. (Convert the surface integral to a polar integral)



$$\iint_S 1\, d\sigma = \iint_R 1 \cdot \frac{|\nabla g|}{|\nabla g \cdot p|}\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{2} 1 \cdot \underbrace{\sqrt{4r^2+1}}_{u} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_1^{17} \frac{\sqrt{u}}{8}\, du\, d\theta$$

$$= 2\pi \cdot \frac{1}{8} \cdot \frac{17^{3/2} - 1^{3/2}}{3/2}$$

$$= \frac{\pi}{6}\left(17^{3/2} - 1\right)$$

$$g(x,y,z) = z - 9 + x^2 + y^2$$

$$\nabla g = \langle 2x, 2y, 1\rangle$$

$$|\nabla g| = \sqrt{4x^2 + 4y^2 + 1}$$

$$|\nabla g \cdot p| = 1$$

12. [16 points] Use Stokes' theorem to convert $\iint_S (\nabla \times \langle y, -z, x\rangle) \cdot \mathbf{n}\, d\sigma$ to a line integral and compute its value.
S is the hemisphere given by $x^2 + y^2 + z^2 = 9$ and $z \geq 0$ and $\mathbf{n}$ is the unit outward normal to S. The curve C can be written as $\mathbf{r}(t) = \langle \underbrace{3\cos(t)}_{x(t)}, \underbrace{3\sin(t)}_{y(t)}, \underbrace{0}_{z(t)} \rangle$.

$$\iint_S (\nabla \times F) \cdot n\, d\sigma$$

|| Stokes' theorem

$$r'(t) = \langle -3\sin t, 3\cos t, 0\rangle$$

$$\oint_C F \cdot dr = \oint_C \langle y, -z, x\rangle \cdot dr = \oint_C \langle 3\sin t, 0, 3\cos t\rangle \cdot r'(t)\, dt$$

$$= \int_{t=0}^{2\pi} -9\sin^2 t\, dt = -9\int_0^{2\pi} \frac{1-\cos(2t)}{2}\, dt = -9 \cdot \frac{2\pi}{2}$$

$$= -9\pi$$