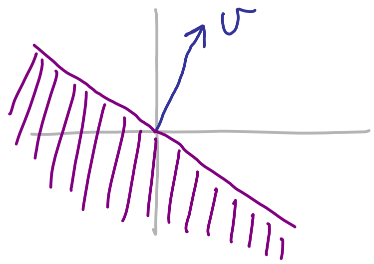
Math 234 HW 1 Solutions

12.3.29

ⓐ $|\cos\theta| \leq 1$, hence $\underbrace{|u|\cdot|v|\cdot|\cos\theta|}_{|u\cdot v|} \leq |u|\cdot|v|$

ⓑ $|u\cdot v| = |u|\cdot|v|$ if at least one of the vectors is zero or if $|\cos\theta| = 1$, i.e., u and v are parallel.

12.3.30



$(xi+yj)\cdot v = 0$ means $\langle x,y\rangle \perp v$
this specifies a line whose normal is v.

$(xi+yj)\cdot v \leq 0$ corresponds to the region which is the half plane in the direction $-v$.

12.4.24

$u = \langle 1, 2, -1\rangle$
$v = \langle -1, 1, 1\rangle$
$w = \langle 1, 0, 1\rangle$
$r = \frac{\pi}{2}\langle -1, -2, 1\rangle$

ⓐ perpendicular: $u\cdot v = 0$
$r\cdot v = 0$
$v\cdot w = 0$
$u\cdot w = 0$
$r\cdot w = 0$

ⓑ $u \parallel r$: $-\frac{\pi}{2}\cdot u = r$

12.4.30 u, v, w nonzero

ⓐ a vector orthogonal to $u\times v$ and $u\times w$: u

ⓑ a vector orthogonal to $u+v$ and $u-v$: $(u+v)\times(u-v)$

$= u\times(u-v) + v\times(u-v)$

$= \underbrace{u\times u}_{0} - u\times v + v\times u - \underbrace{v\times v}_{0} = -u\times v - u\times v = -2u\times v$

Remark: since $u+v$ and $u-v$ lie on the plane spanned by u and v, $u\times v$ is orthogonal to both.

ⓒ a vector of length $|u|$ in the direction of v : $|u|\cdot\frac{v}{|v|}$

ⓓ the area of the parallelogram determined by u and w: $|u\times w|$

12.5.27

$$x = 2t+1 \qquad x = s+2$$
$$y = 3t+2 \qquad y = 2s+4$$
$$z = 4t+3 \qquad z = -4s-1$$

Point of intersection: same x: $2t+1 = s+2 \rightarrow s = 2t-1$

same y: $3t+2 = 2s+4$
$= 2(2t-1)+4$
$3t+2 = 4t+2 \longrightarrow t=0 \quad s=-1$

verify: same z? $0+3 = -4(-1)-1$ ✓

Hence $(1,2,3)$ is the common point.

The plane determined by these lines goes through $(1,2,3)$ and its normal is orthogonal to both direction vectors $\langle 2,3,4\rangle$ and $\langle 1,2,-4\rangle$

$$\vec{n} = \langle 2,3,4\rangle \times \langle 1,2,-4\rangle$$
$$= \begin{vmatrix} i & j & k \\ 2 & 3 & 4 \\ 1 & 2 & -4 \end{vmatrix} = i\left(3\cdot(-4)-4\cdot 2\right) - j\left(2\cdot(-4)-4\cdot 1\right) + k\left(2\cdot 2-3\cdot 1\right) = \langle -20,12,1\rangle$$

Equation of plane: $-20(x-1)+12(y-2)+1\cdot(z-3)=0$

12.5.58

$$3x-6y-2z=3 \qquad \vec{n}_1 = \langle 3,-6,-2\rangle$$
$$2x+y-2z=2 \qquad \vec{n}_2 = \langle 2,1,-2\rangle$$

direction of line of intersection: $\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} i & j & k \\ 3 & -6 & -2 \\ 2 & 1 & -2 \end{vmatrix} = \langle 14,2,15\rangle$

We need a common point: find one with $z=0$:

$3x-6y=3$
$2x+y=2 \longrightarrow y = 2-2x \longrightarrow 3x-6(2-2x)=3$
$15x=15$
$x=1$
$\downarrow$
$y=0$

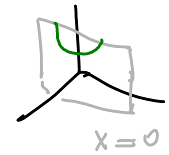
Point: $(1,0,0)$
Direction: $\langle 14,2,15\rangle$
equation: $x = 1+14t, \quad y=2t, \quad z=15t$

12.6.47 Sketch the surface given by $z = 1 + y^2 - x^2$

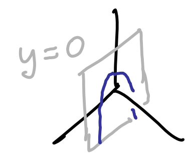
idea: $z = y^2 - x^2$ is the hyperbolic paraboloid,
shift it up in z direction by 1 unit.

x appears in second power only, hence surface has symmetry with respect to the plane $x = 0$ (replace x by $-x$, same eqn.)
Similarly for y.

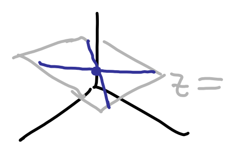
cross sections: $x = 0$: $z = 1 + y^2$ parabola



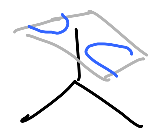
$y = 0$: $z = 1 - x^2$ parabola



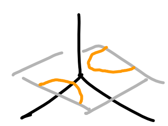
$z = 1$: $1 = 1 + y^2 - x^2 \longrightarrow y = \pm x$

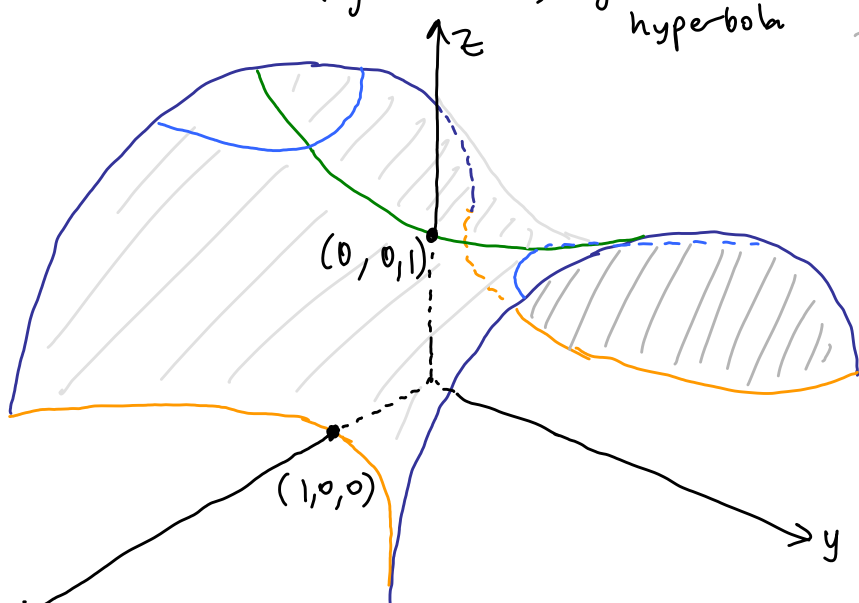


$z = 2$: $2 = 1 + y^2 - x^2 \longrightarrow y^2 = 1 + x^2$ hyperbola



$z = 0$: $0 = 1 + y^2 - x^2 \longrightarrow y^2 = 1 - x^2$ hyperbola





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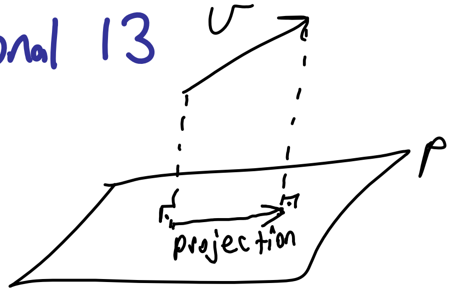


plane

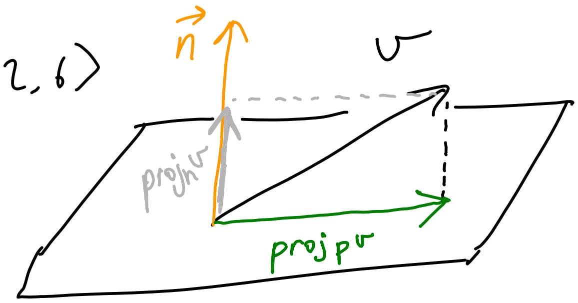
Idea: split v as a sum

$$v = \underset{\substack{\text{"} \\ \text{projection} \\ \text{to } P}}{u} + w$$

Note that w will be orthogonal to the plane P, hence $\vec{w} = \text{proj}_{\vec{n}} \vec{v}$ where $\vec{n}$ is the normal of the plane P.

$P: \quad x + 2y + 6z = 6 \qquad \vec{n} = \langle 1,2,6 \rangle$

$v: \quad \langle 1,1,1 \rangle$



$$\text{proj}_n v = \frac{v \cdot n}{|n|} \cdot \frac{n}{|n|} = \frac{1}{|n|^2} \cdot \left(\langle 1,1,1 \rangle \cdot \langle 1,2,6 \rangle \right) \cdot \langle 1,2,6 \rangle$$

$$= \frac{9}{41} \langle 1,2,6 \rangle$$

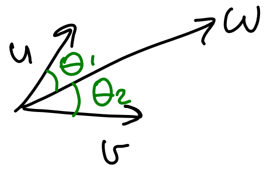
Hence $\text{Proj}_P v = v - \text{Proj}_n v = \langle 1,1,1 \rangle - \frac{9}{41} \langle 1,2,6 \rangle$

Additional 22

Show that $w = |v|u + |u|v$ bisects the angle between u and v.

① u, v span a plane P, and since w is a linear combination of u and v, w lies in P, too.

②



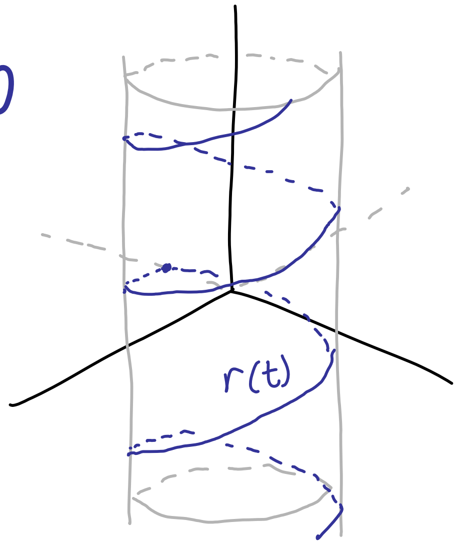
$$\cos\theta_1 = \frac{u \cdot w}{|u| \cdot |w|} = \frac{u \cdot (|v|u) + u \cdot (|u|v)}{|u| \cdot |w|}$$

$$\cos\theta_2 = \frac{v \cdot w}{|v| \cdot |w|} = \frac{v \cdot (|v|u) + v \cdot (|u|v)}{|v| \cdot |w|}$$

is $\cos\theta_1 = \cos\theta_2$? use $u \cdot u = |u|^2$, $v \cdot v = |v|^2$, $u \cdot v = v \cdot u$

$$\left.\begin{aligned} \cos\theta_1 &= \frac{|u||v| + u \cdot v}{|w|} \\ \cos\theta_2 &= \frac{v \cdot u + |u| \cdot |v|}{|w|} \end{aligned}\right\} \text{equal.}$$

13.3.10



at $t=0$, $\vec{r}(0) = (0,-12,0)$
Find location 13π before $t=0$.

① how long did it take to travel 13π?

solve $\displaystyle\int_{t_0}^{0} |v|\,ds = 13\pi$ ☆

$t_0 < 0$ since we need direction opposite to arclength.

$r(t) = \langle 12\sin t, -12\cos t, 5t \rangle$ $\quad$ $v(t) = \langle 12\cos t, 12\sin t, 5 \rangle$

$$\int_{t_0}^{0} \left| \sqrt{12^2\cos^2 s + 12^2\sin^2 s + 5^2} \right| ds = \int_{t_0}^{0} |13|\,ds = 0 - 13t_0$$

hence solving ☆, we get $-13t_0 = 13\pi$, $t_0 = -\pi$

② Location: $\vec{r}(-\pi) = \langle 0, 12, -5\pi \rangle$