234.201 HW 2 Solutions

14.4.38

$z = \ln(q)$, $\quad q = \sqrt{v+3}\,\tan^{-1}(u)$

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$ when $u=1$, $v=-2$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial q}\cdot\frac{\partial q}{\partial u} = \frac{1}{q}\cdot\sqrt{v+3}\cdot\frac{1}{1+u^2}$$

$$\left.\frac{\partial z}{\partial u}\right|_{u=1,v=-2} = \frac{1}{1\cdot\frac{\pi}{4}}\cdot 1\cdot\frac{1}{2} = \frac{2}{\pi}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial q}\cdot\frac{\partial q}{\partial v} = \frac{1}{q}\cdot\frac{1}{2\sqrt{v+3}}\cdot\tan^{-1}(u)$$

$$\left.\frac{\partial z}{\partial v}\right|_{u=1,v=-2} = \frac{1}{1\cdot\pi/4}\cdot\frac{1}{2\cdot 1}\cdot\frac{\pi}{4} = \frac{1}{2}$$

14.4.42

$w = f(x,y)$, $\quad x = r\cos\theta$, $\quad y = r\sin\theta$

(a)
$$\frac{\partial w}{\partial r} = \underbrace{\frac{\partial w}{\partial x}}_{= f_x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = f_x\cos\theta + f_y\sin\theta$$

$$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x\cdot(-\sin\theta)\cdot r + f_y\cdot(\cos\theta)\cdot r$$

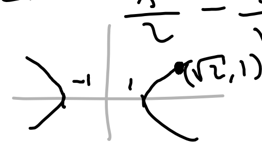
(c)
$$\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2 = (f_x)^2\cos^2\theta + 2f_xf_y\cos\theta\sin\theta + (f_y)^2\sin^2\theta$$
$$+ (f_x)^2\sin^2\theta - 2f_xf_y\cos\theta\sin\theta + (f_y)^2\cos^2\theta$$
$$= \left((f_x)^2 + (f_y)^2\right)\left(\cos^2\theta + \sin^2\theta\right) = (f_x)^2 + (f_y)^2$$

14.5.4

$g(x,y) = \frac{x^2}{2} - \frac{y^2}{2}$, find $\nabla g|_{(\sqrt{2},1)}$ and draw level curve through $(\sqrt{2},1)$.

$$\nabla g = \langle x, -y\rangle, \qquad \nabla g|_{(\sqrt{2},1)} = \langle \sqrt{2}, -1\rangle$$

$g(\sqrt{2},1) = 1 - \frac{1}{2} = 1/2$ $\qquad$ level curve: $\frac{x^2}{2} - \frac{y^2}{2} = \frac{1}{2}$, i.e. $x^2 - y^2 = 1$

hyperbola 

14.5.16 $h(x,y,z) = \cos xy + e^{yz} + \ln zx$, $P=(1,0,1/2)$

Find derivative of h in the direction $A = i+2j+2k$ at P.

$$u = \text{unit vector in direction of } A = \frac{A}{|A|} = \frac{\langle 1,2,2\rangle}{\sqrt{1+2^2+2^2}} = \frac{1}{3}\langle 1,2,2\rangle$$

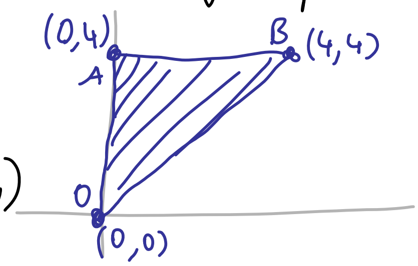
$$\nabla h = \langle h_x, h_y, h_z\rangle =$$

$$= \left\langle -y\sin xy + \frac{z}{zx},\ -x\sin xy + ze^{yz},\ ye^{yz} + \frac{x}{zx}\right\rangle$$

$$\nabla h|_P = \langle 1, \tfrac{1}{2}, 2\rangle$$

directional derivative $= (D_u h)_P = \langle 1, \frac{1}{2}, 2\rangle \cdot \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3}\rangle = 2$

14.7.32 Find the absolute minima and maxima of $D(x,y) = x^2 - xy + y^2 + 1$ on the closed triangular plate



interior:

$D_x = 2x - y$ $\qquad D_x = D_y = 0$

$D_y = -x + 2y$ $\qquad \longrightarrow x = y = 0$ (on boundary)

hence no local min/max inside the triangle.

boundary: $D(0,0) = \underline{1}$ $\qquad D(0,4) = \underline{17}$ $\qquad D(4,4) = \underline{17}$

on OA: $D(0,y) = y^2 + 1$ increasing for $y \geq 0$, so no crit. pts inside the line segment OA

on AB: $D(x,4) = x^2 - 4x + 17 \xrightarrow{\text{derivative}} 2x - 2 = 0$, $x = 1$

$D(1,4) = 1 - 4 + 16 + 1 = \underline{14}$ (minimum along AB)

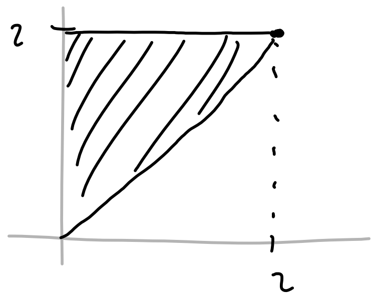
on OB: $D(x,x) = x^2 + 1$ (similar to OA)

Hence absolute max $= 17$ at $(0,4)$ and $(4,4)$, abs min $= 1$ at $(0,0)$

14.7.51 $f(x,y) = x+y$ doesn't have an absolute max on the closed region $x \geq 0$, $y \geq 0$.
This doesn't contradict the text because the region is not bounded.

15.1.32 $$\int_{x=0}^{2} \int_{y=x}^{2} 2y^2 \sin(xy)\, dy\, dx$$

Sketch region of integration, reverse the order of integration, evaluate.



$$\int_{y=0}^{2} \int_{x=0}^{y} 2y^2 \sin(xy)\, dx\, dy$$

$$= \int_{y=0}^{2} 2y^2 \cdot \left(\frac{-1}{y} \cos(xy) \right)\Bigg|_{x=0}^{y} dy$$

$$= \int_{y=0}^{2} 2y^2 \cdot \left(-\frac{1}{y} \cos(y^2) - \frac{-1}{y} \cos(0) \right) dy = \int_{y=0}^{2} -2y\cos(y^2)\, dy + \int_{y=0}^{2} 2y\, dy$$

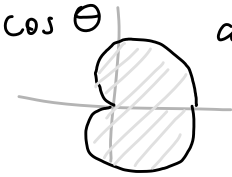
$u = y^2$
$du = 2y\, dy$

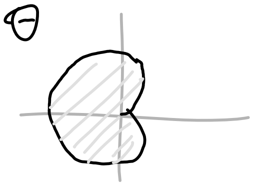
$$= \int_{u=0}^{4} -\cos u\, du + y^2\Big|_0^2 = -\sin(u)\Big|_0^4 + 4 = -\sin(4) + 4$$

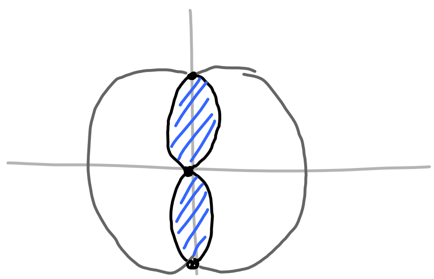
15.1.62 Which region minimizes $\iint_R (x^2+y^2-9)\, dA$?

The integrand is negative on $x^2+y^2<9$, so the integral would be minimum on $R = \{(x,y) \mid x^2+y^2 \leq 9\}$, which is the disk of radius 3 centered at the origin.
On any smaller region, the value of the integral is closer to zero, hence bigger, and on any region containing R, points outside R contribute positively to the integral.

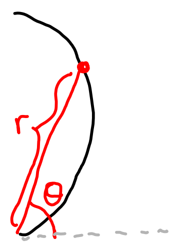
15.3.22 Find the area of the region common to the interiors of $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$







By symmetry, we'll find  and multiply by 4.



θ ranges from 0 to $\pi/2$, r is given by $1-\cos\theta$

$$\text{area} = 4\cdot\int_{\theta=0}^{\pi/2}\int_{r=0}^{1-\cos\theta} r\,dr\,d\theta$$

$$= 4\int_{\theta=0}^{\pi/2} \frac{r^2}{2}\Big|_0^{1-\cos\theta} d\theta = \frac{4}{2}\int_{\theta=0}^{\pi/2} 1 - 2\cos\theta + \cos^2\theta \, d\theta$$

$$= 2\int_0^{\pi/2} 1\,d\theta - 4\int_0^{\pi/2}\cos\theta\,d\theta + 2\int_0^{\pi/2}\frac{1+\cos 2\theta}{2}\,d\theta$$

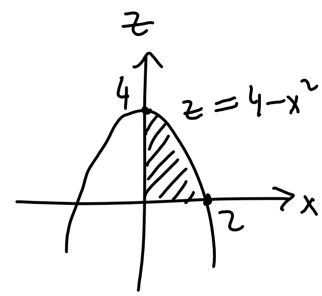
$$= 2\cdot\frac{\pi}{2} - 4\cdot\left(\sin\frac{\pi}{2} - \sin 0\right) + \left(\frac{\pi}{2} + \frac{\sin(\pi)}{2} - 0 + \frac{\sin 0}{2}\right)$$

$$= \pi - 4 + \pi/2 = \frac{3\pi}{2} - 4$$

15.4.44 Change the order of integration and evaluate:

$$\int_{x=0}^{2}\int_{z=0}^{4-x^2}\underbrace{\int_{y=0}^{x}\frac{\sin 2z}{4-z}\,dy}_{\text{easy to integrate}}\,dz\,dx$$

easy to integrate, so switch x and z:



$$\int_{z=0}^{4}\int_{x=0}^{\sqrt{4-z}}\int_{y=0}^{x}\frac{\sin 2z}{4-z}\,dy\,dx\,dz = \dots = \int_0^4 \frac{1}{2}\sin 2z\,dz = -\frac{1}{4}(\cos 8 - 1)$$