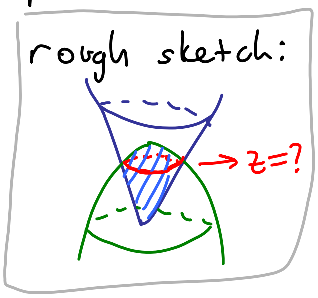
Math 234 Quiz 3 Solutions August 6, 2007

(1) Using cylindrical coordinates, find the volume of the region bounded below by the cone $z=\sqrt{x^2+y^2}$ and above by the paraboloid $z = 2 - x^2 - y^2$



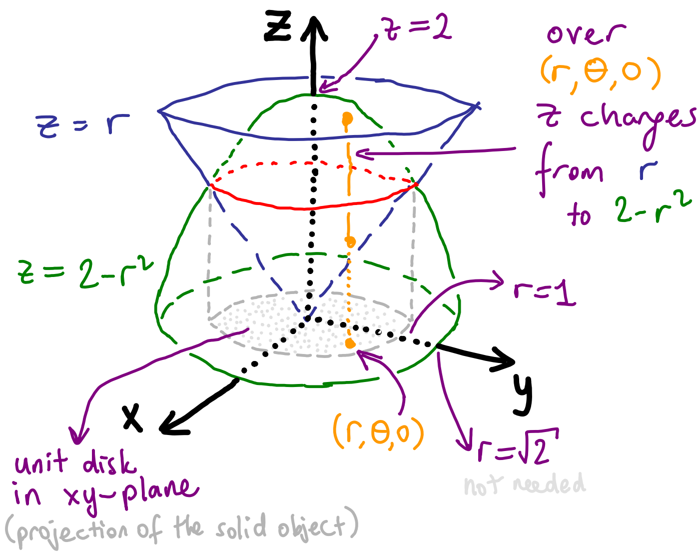
They intersect at $z=1$:

$$\sqrt{x^2+y^2} = 2 - x^2 - y^2$$

$$\sqrt{r^2} = 2 - r^2 \longrightarrow r^2 + r - 2 = 0$$

$$(r-1)(r+2) = 0$$

$$z = \sqrt{r^2} = 1 \longleftarrow r=1, \quad \cancel{r=-2}$$



$$\text{Volume} = \underbrace{\int_{\theta=0}^{2\pi}\int_{r=0}^{1}}_{\text{unit disk}}\int_{z=r}^{2-r^2} 1 \underbrace{dV}_{r\,dz\,dr\,d\theta}$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cdot z\Big|_{z=r}^{2-r^2} dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} \underbrace{r\cdot(2-r^2-r)}_{2r-r^2-r^3}dr\,d\theta = \int_{\theta=0}^{2\pi} r^2 - \frac{r^3}{3} - \frac{r^4}{4}\Big|_{r=0}^{1} d\theta = 2\pi\cdot\left(1-\frac{1}{3}-\frac{1}{4}\right)$$

$$= 2\pi\cdot\frac{5}{12} = \frac{5\pi}{6}$$

(2) Evaluate the line integral $\int_C (x+y-z)\,ds$ along the line segment C between $(1,0,1)$ and $(1,1,0)$.

$$r(t) = (1,\ t,\ 1-t),\quad 0\le t\le 1$$

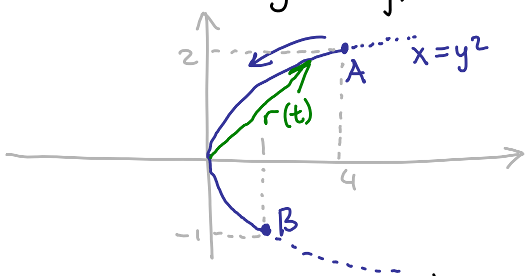
$$r'(t) = (0,\ 1,\ -1)$$

$$|r'(t)| = \sqrt{2}$$

$$\int_C f(x,y,z)\,ds = \int_{t=a}^{b} f(r(t))\underbrace{\left|\frac{dr}{dt}\right| dt}_{ds}$$

$$= \int_0^1 (\underset{1}{\underset{\parallel}{x}} + \underset{t}{\underset{\parallel}{y}} - \underset{1-t}{\underset{\parallel}{z}})\cdot\sqrt{2}\,dt = \int_0^1 2t\sqrt{2}\,dt = 2\sqrt{2}\cdot\frac{t^2}{2}\Big|_0^1 = \sqrt{2}$$

③ Find the work done by the vector field $\vec{F} = \langle x^2, -y \rangle$ along the curve $x = y^2$ from $(4,2)$ to $(1,-1)$.



(t^2, t) parametrizes the curve, but in the opposite direction, hence
$r(t) = \left((-t)^2, -t\right) = (t^2, -t)$

$$\text{Work} = \int_C \vec{F}\cdot\vec{T}\,ds = \int_{t=a}^{b} \vec{F}\cdot\frac{d\vec{r}}{dt}\,dt = \int_{t=-2}^{1} \langle \underset{\underset{t^4}{\parallel}}{x^2}, \underset{\underset{t}{\parallel}}{-y} \rangle \cdot \underbrace{\langle 2t, -1 \rangle}_{r'(t)}\,dt$$

$$= \int_{t=-2}^{1} 2t^5 - t\,dt = 2\cdot\frac{t^6}{6} - \frac{t^2}{2}\Big|_{t=-2}^{1} = \left(\frac{1}{3} - \frac{1}{2}\right) - \left(\frac{64}{3} - \frac{4}{2}\right) = \frac{-1}{6} - \frac{116}{6} = \frac{-117}{6}$$

④ Show that the vector field $\vec{F} = \langle 2xz, \sin(y), x^2 \rangle$ is a conservative vector field, find a potential function and compute the integral $\int_C \vec{F}\cdot d\vec{r}$ from $(1,0,-1)$ to $(2,\pi,3)$.

$M = 2xz$
$N = \sin(y)$
$P = x^2$

$$\left.\begin{aligned} M_y &= 0 = N_x = 0 \\ M_z &= 2x = P_x = 2x \\ N_z &= 0 = P_y = 0 \end{aligned}\right\}$$

hence by component test $\vec{F}$ is conservative, so $\vec{F} = \nabla f$ for some $f(x,y,z)$

$$f_x = M \longrightarrow \frac{\partial f}{\partial x} = 2xz \longrightarrow f(x,y,z) = \int 2xz\,dx = x^2 z + g(y,z)$$

$$\underset{\underset{0+\frac{\partial g}{\partial y}}{\parallel}}{f_y} = N = \sin(y) \longrightarrow g(y,z) = \int \sin(y)\,dy = -\cos(y) + h(z)$$

So far: $f(x,y,z) = x^2 z - \cos y + h(z)$

$$\underset{\underset{x^2 + \frac{dh}{dz}}{\parallel}}{f_z} = P = x^2 \longrightarrow \frac{dh}{dz} = 0, \quad h(z) = c \quad \text{(constant)}$$

hence $f(x,y,z) = x^2 z - \cos y + c$

$$\int_A^B \vec{F}\cdot d\vec{r} = \int_A^B \nabla f\cdot d\vec{r} = \underset{\underset{(2,\pi,3)}{\parallel}}{f(B)} - \underset{\underset{(1,0,-1)}{\parallel}}{f(A)} = (4\cdot 3 - \cos\pi) - (1\cdot(-1) - \cos 0)$$
$$= 12 - (-1) - (-1-1) = 15$$