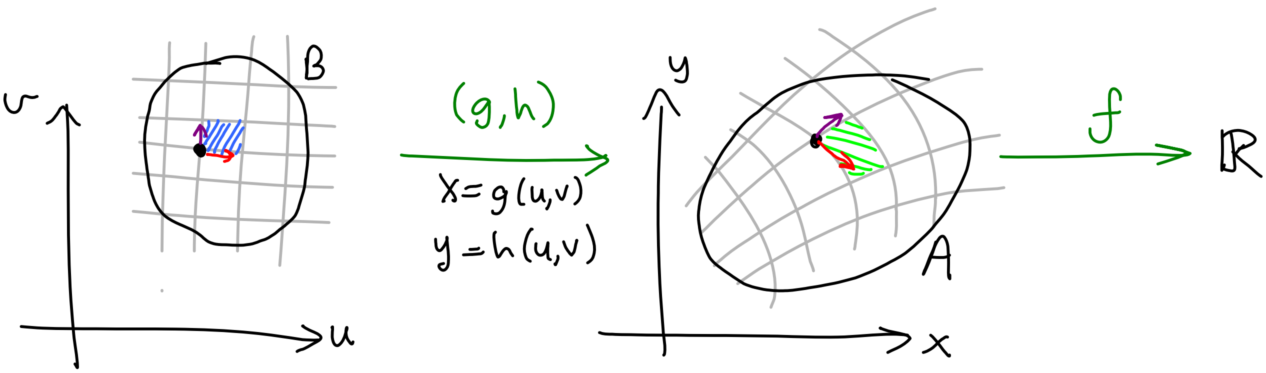
Substitution in Double integrals

In one variable: if $g:[a,b] \longrightarrow [c,d]$ is one-to-one and onto and continuously diff'ble

$$\int_c^d f(x)\,dx = \int_a^b f(g(u))\,|g'(u)|\,du$$

What is the corresponding formula for two variables?

if $\begin{array}{l} x = g(u,v) \\ y = h(u,v) \end{array}$ let $J(u,v) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}$ the Jacobian matrix of the coordinate transformation



Partition A into a curved grid which is the image of usual grid in B. Let (u_k, v_k) be the grid points in B, and (x_k, y_k) the corresponding points in A. Then

$$\iint_A f(x,y)\,dx\,dy \approx \sum f(x_k, y_k) \cdot \underbrace{\text{area}\begin{pmatrix} k^{th} \\ \text{curved} \\ \text{region}\end{pmatrix}}_{\approx}$$

explained in next page

$$\approx \sum f\left(g(u_k,v_k),\, h(u_k,v_k)\right) \cdot \overbrace{\left|\frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v}\right| \cdot \text{area}\begin{pmatrix} k\text{th} \\ \text{rectangular} \\ \text{region}\end{pmatrix}}$$

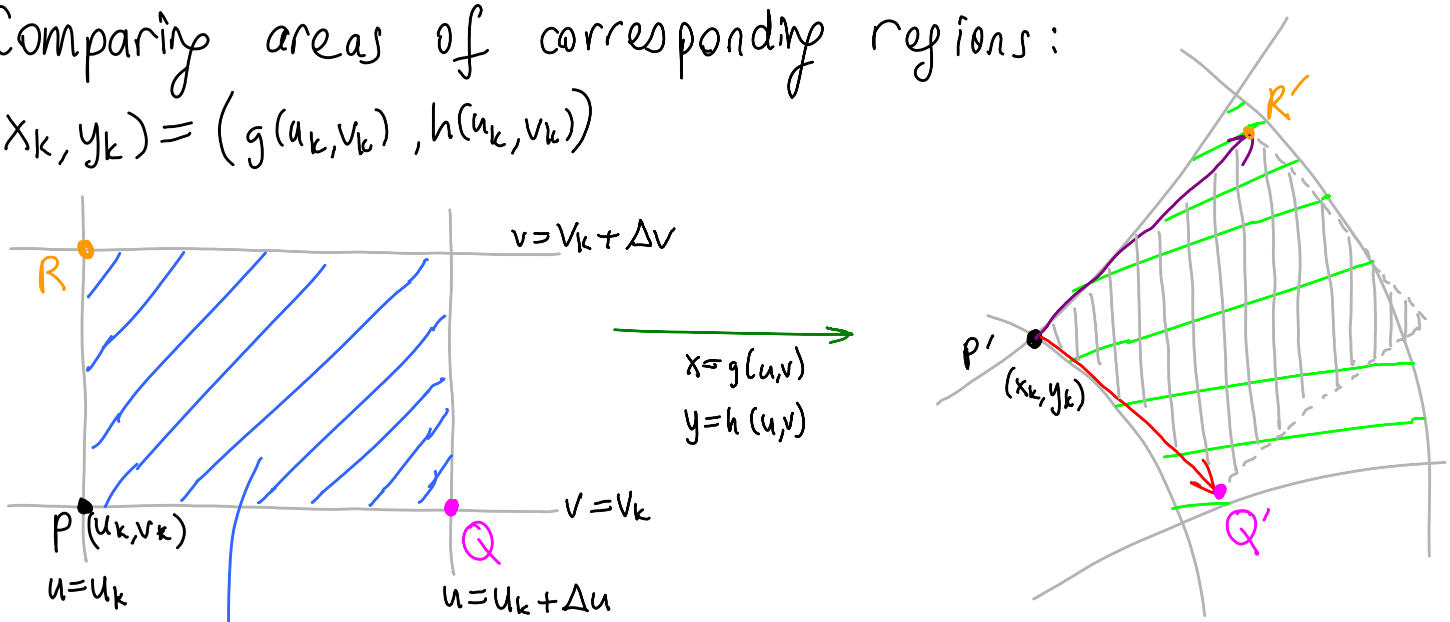
$$\approx \iint_B f\left(g(u,v),\, h(u,v)\right) \cdot \Big|\underbrace{|\,J(u,v)\,|}_{\text{Jacobian determinant}}\Big|\,du\,dv$$

As the partition gets finer we get the equality in the limit:

$$\iint_A f(x,y)\,dx\,dy = \iint_B f(g(u,v),\, h(u,v))\,\|J(u,v)\|\,du\,dv$$

Comparing areas of corresponding regions:

$P' = (x_k, y_k) = \left(g(u_k,v_k),\, h(u_k,v_k)\right)$



$$\Delta A = \Delta u \cdot \Delta v$$

Linear approximation for Q':

$$g(u_k + \Delta u, v_k) \approx \underbrace{g(u_k, v_k)}_{x_k} + \underbrace{\frac{\partial g}{\partial u}}_{\frac{\partial x}{\partial u}} \cdot \Delta u$$

$$h(u_k + \Delta u, v_k) \approx \underbrace{h(u_k, v_k)}_{y_k} + \underbrace{\frac{\partial h}{\partial u}}_{\frac{\partial y}{\partial u}} \cdot \Delta u$$

We get $\overrightarrow{P'Q'} = \left\langle \frac{\partial x}{\partial u}\Delta u, \frac{\partial y}{\partial u}\Delta u \right\rangle$

Similarly $\overrightarrow{P'R'} = \left\langle \frac{\partial x}{\partial v}\Delta v, \frac{\partial y}{\partial v}\Delta v \right\rangle$

The green area is roughly the same as the area of the paralelogram spanned by $P'Q'$ and $P'R'$:

$$|P'Q' \times P'R'| = \left\| \begin{matrix} i & j & k \\ \frac{\partial x}{\partial u}\Delta u & \frac{\partial y}{\partial u}\Delta u & 0 \\ \frac{\partial x}{\partial v}\Delta v & \frac{\partial y}{\partial v}\Delta v & 0 \end{matrix} \right\| = \left| \frac{\partial x}{\partial u}\cdot\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\cdot\frac{\partial x}{\partial v} \right| \cdot \underbrace{\Delta u \cdot \Delta v}_{\text{area of blue region}}$$

determinant

absolute value

$$= \big|\, |J(u,v)| \,\big| \cdot \Delta u \cdot \Delta v$$

Remark: $|J(u,v)| = \dfrac{1}{|J(x,y)|}$

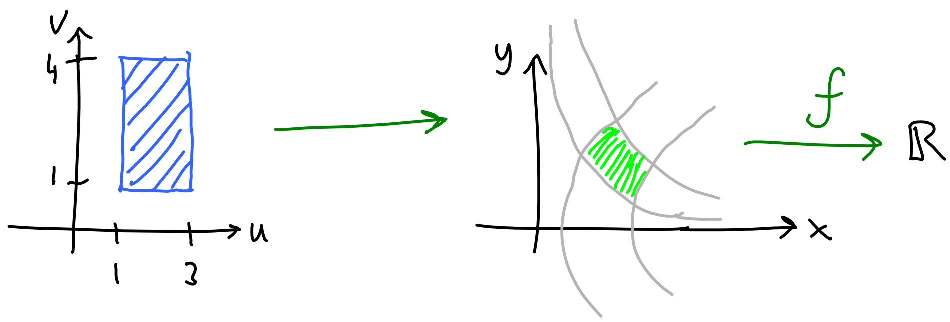
Similar rule holds for other multiple integrals (triple, quadruple, ...)

Example Compute $\iint_R (x^2+y^2)\,dx\,dy = ?$ where R is the region $1 \leq xy \leq 3$, $1 \leq x^2-y^2 \leq 4$

$u = xy$

$v = x^2 - y^2$

solving for x,y difficult, so use:



$$|J(u,v)| = \frac{1}{|J(x,y)|}$$

$$J(x,y) = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix} = \begin{pmatrix} y & x \\ 2x & -2y \end{pmatrix}$$

$$||J(x,y)|| = |-2y^2 - 2x^2| = 2x^2 + 2y^2$$

$$||J(u,v)|| = \frac{1}{2x^2+2y^2} \qquad \text{hence} \quad dx\,dy = \frac{1}{2(x^2+y^2)}\,du\,dv$$

$$\iint_R (x^2+y^2)\,dx\,dy = \int_{v=1}^{4}\int_{u=1}^{3} (x^2+y^2)\cdot\frac{1}{2(x^2+y^2)}\,du\,dv = \int_{v=1}^{4}\int_{u=1}^{3} \frac{1}{2}\,du\,dv$$

$$= \frac{1}{2}\underbrace{\int_{v=1}^{4}\int_{u=1}^{3} du\,dv}_{\text{area of rectangle}} = \frac{1}{2}\cdot 3\cdot 2 = 3$$

Try #9 page 1136.