Sequences and the Difference Operator

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1 Sequences

Number sequences appear all throughout mathematics. Examples are:

- $1, 2, 3, 4, 5, 6, \ldots$ (the natural numbers)
- $1, 4, 9, 16, 25, 26, \ldots$ (the square numbers)
- $3, 9, 27, 81, 243, 729, \ldots$ (the powers of 3)
- $1, 2, 3, 5, 8, 13, \ldots$ (the Fibonacci numbers)
- $1, 2, 4, 7, 12, 20, \ldots$ (shifted Fibonacci numbers)
- $1, 2, 8, 42, 262, 1828, \ldots$ (the meandric numbers)

Often we give the sequence a name or label so that we can easily reference it later. For instance, we may decide that a_n will represent the nth natural number. That is,

$$a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 4, a_5 = 5, a_6 = 6, \ldots$$

and in general

$$a_n = n.$$

Let b_n represent the nth square number. Then

$$b_1 = 1, b_2 = 4, b_3 = 9, b_4 = 16, b_5 = 25, b_6 = 36, \ldots$$

and in general

$$b_n = n^2.$$

Let c_n represent the nth power of 3. Then

$$c_1 = 3, c_2 = 9, c_3 = 27, c_4 = 81, c_5 = 243, c_6 = 729, \ldots$$

and in general

$$c_n = 3^n.$$

Notice that, using this formula for c_n, it makes sense to talk about c_0, that is, the 0th power of 3:

$$c_0 = 3^0 = 1.$$

The same is true for a_n and b_n:

$$a_0 = 0 \text{ and } b_0 = 0^2 = 0.$$

These sequences are nice because we can write down a formula for the nth element in the sequence. That is, given the formula I can immediately calculate the nth element of the sequence. For instance, if I want to know the 277th square number, I simply compute

$$b_{277} = 277^2 = 76729$$

or if I want to know the 13th power of 3, I compute

$$c_{13} = 3^{13} = 1594323$$

The Fibonnaci numbers are a little different. I begin by giving them a name, say f_n for the nth Fibonacci number. Then

$$f_1 = 1, f_2 = 2, f_3 = 3, f_4 = 5, f_5 = 8, f_6 = 13, \ldots$$

But how do I write down a formula for this sequence?!? Right now it is not obvious. However, it is obvious that

$$f_1 = 1, f_2 = 2, \text{ and } f_n = f_{n-1} + f_{n-2} \text{ for } n > 2.$$

This is not a formula in the sense that I can immediately calculate the nth Fibonacci number from the sequence – the calculation requires knowledge of the previous two Fibonacci numbers. The equation $f_n = f_{n-1} + f_{n-2}$ is a called a *recurrence relation* for the Fibonacci numbers. In a sense, it is the next best thing to a formula – given the first couple of terms of the sequence, I can compute the rest rather quickly.

One of the things we will explore in these lectures is the following: given a recurrence relation for a sequence, can we derive its formula? The answer is sometimes yes. In the case of Fibonacci numbers, it turns out that

$$f_n = \frac{\left(1+\sqrt{5}\right)^{n+1} - \left(1-\sqrt{5}\right)^{n+1}}{2^{n+1}\sqrt{5}}.$$

Call the nth shifted Fibonacci number g_n. Then g_n satisfies the recurrence relation

$$g_1 = 1, g_2 = 2, \text{ and } g_n = g_{n-1} + g_{n-2} + 1 \text{ for } n > 2$$

and the formula

$$g_n = \frac{\left(3+\sqrt{5}\right)\left(1+\sqrt{5}\right)^{n} - \left(3-\sqrt{5}\right)\left(1-\sqrt{5}\right)^{n}}{2^{n+1}\sqrt{5}} - 1.$$

The last sequence is very different from all the others. Let m_n be the nth meandric number. Then

$$m_1 = 1, m_2 = 2, m_3 = 8, m_4 = 42, m_5 = 262, m_6 = 1828, \ldots$$

As of today, there is no formula or recurrence relation to describe this sequence – but maybe you'll figure one out!

1.1 Exercises

Write down a formula and/or a recurrence relation for the following sequences.

1. $23, 23, 23, 23, 23, 23, \ldots$

2. $0, 1, 8, 27, 64, 125, 216, \ldots$

3. $7, 8, 15, 34, 71, 132, 223, \ldots$

4. $0, 1, 0, -1, 0, 1, 0, -1, \ldots$

5. $1, 2, 6, 24, 120, 720, \ldots$

6. $1, 7, 19, 37, 61, 91, \ldots$

1.2 Solutions

1. $23, 23, 23, 23, 23, 23, \ldots$

   $a_n = 23; \qquad a_1 = 23, a_n = a_{n-1}$ for $n > 1$

2. $0, 1, 8, 27, 64, 125, 216, \ldots$

   $b_n = n^3; \qquad b_0 = 0, b_{n+1} = b_n + z_n$

3. $7, 8, 15, 34, 71, 132, 223, \ldots$

   $c_n = b_n + 7 = n^3 + 7; \qquad c_0 = 7, c_{n+1} = c_n + z_n$

4. $0, 1, 0, -1, 0, 1, 0, -1, \ldots$

   $x_n = \sin\left(n\frac{\pi}{2}\right)$

5. $1, 2, 6, 24, 120, 720, \ldots$

   $y_n = n!$

6. $1, 7, 19, 37, 61, 91, \ldots$

   $z_n = b_{n+1} - b_n = 3n^2 + 3n + 1; z_0 = 1, z_n = z_{n-1} + 6n$ for $n > 1$

2 The Difference Operator

How did the exercises go? My guess is that you were able to write down formulae for exercises 1 through 5 quickly, but only came up with a recurrence relation for exercise 6. What is hard about exercise 6? You have probably seen the other five sequences before (or at least sequences very similar) and as a result you already have a good intuition for the numbers appearing in these sequences. But the sequence in exercise 6 is not so common. Let's look at it again:

$$1, 7, 19, 37, 61, 91, \ldots$$

Let's call the nth term in this sequence z_n (here we will assume $z_0 = 1, z_1 = 7$, etc.). Is this sequence related to any of the other sequences from the exercises?

Surprisingly (or not), it is related to the sequence in exercise 2. Let b_n be this sequence. We can easily spot the formula for b_n:

$$b_n = n^3.$$

Now what was the recurrence relation you came up with for b_n? One recurrence relation is

$$b_0 = 0, \; b_{n+1} = b_n + z_n \qquad (n \geq 0).$$

Another way to look at this is that the difference between the $(n+1)$th and nth element of the sequence b_n is the nth element in the sequence z_n:

$$b_{n+1} - b_n = z_n \qquad (n \geq 0).$$

Therefore, the formula for z_n is

$$\begin{aligned} z_n &= b_{n+1} - b_n \\ &= (n+1)^3 - n^3 \\ &= n^3 + 3n^2 + 3n + 1 - n^3 \\ &= 3n^2 + 3n + 1. \end{aligned}$$

We have a name for this relationship between the sequences b_n and z_n. We say that the sequence z_n is the *difference* of the sequence b_n and write

$$\Delta b_n \equiv b_{n+1} - b_n = z_n$$

and call Δ the *difference operator.*

Now notice something else. Let c_n represent the nth element of the sequence in exercise 3. It is clear that

$$c_n = n^3 + 7.$$

What is Δc_n?

$$\begin{aligned} \Delta c_n &= c_{n+1} - c_n \\ &= (n+1)^3 + 7 - (n^3 + 7) \\ &= n^3 + 3n^2 + 3n + 1 + 7 - n^3 - 7 \\ &= 3n^2 + 3n + 1 \\ &= z_n. \end{aligned}$$

Wow! The difference of c_n is again z_n. At first this may seem surprising, but actually it is quite obvious given the relationship between b_n and c_n.

What are the differences of the other sequences in the exercises? Let a_n be the sequence in exercise 1. Then $a_n = 23$ and

$$\Delta a_n = a_{n+1} - a_n = 23 - 23 = 0.$$

Let x_n be the sequence in exercise 4. Then $x_n = \sin(n\pi/2)$ and

$$\begin{aligned}
\Delta x_n &= x_{n+1} - x_n \\
&= \sin\left((n+1)\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right) \\
&= \sin\left(n\frac{\pi}{2} + \frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right) \\
&= \sin\left(n\frac{\pi}{2}\right)\cos\left(\frac{\pi}{2}\right) - \cos\left(n\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right) \\
&= \sin\left(n\frac{\pi}{2}\right)\cdot 0 - \cos\left(n\frac{\pi}{2}\right)\cdot 1 - \sin\left(n\frac{\pi}{2}\right) \\
&= -\cos\left(n\frac{\pi}{2}\right) - \sin\left(n\frac{\pi}{2}\right).
\end{aligned}$$

Using this formula to compute the first few elements of the difference sequence gives us

$$\Delta x_0 = -1, \Delta x_1 = -1, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = -1, \Delta x_5 = -1, \Delta x_6 = 1, \Delta x_7 = 1, \ldots$$

which is exactly what we expect.

Let y_n represent the nth element in the sequence of exercise 5. Then $y_n = n!$, and

$$\begin{aligned}
\Delta y_n &= y_{n+1} - y_n \\
&= (n+1)! - n! \\
&= (n+1)\cdot n! - n! \\
&= (n+1-1)\cdot n! \\
&= n \cdot n!
\end{aligned}$$

Finally, we know that $z_n = 3n^2 + 3n + 1$. Thus

$$\begin{aligned}
\Delta z_n &= z_{n+1} - z_n \\
&= 3(n+1)^2 + 3(n+1) + 1 - (3n^2 + 3n + 1) \\
&= 3(n^2 + 2n + 1) + 3n + 3 + 1 - 3n^2 - 3n - 1 \\
&= 3n^2 + 6n + 3 + 3n + 3 + 1 - 3n^2 - 3n - 1 \\
&= 6n + 6 \\
&= 6(n+1)
\end{aligned}$$

Isn't this fun?!?

2.1 Exercises

Compute the difference of each of the following sequences.

1. n^4

2. e^n

3. $\ln(n)$

4. $n^{\underline{3}} \equiv \frac{n!}{(n-3)!}$ for $n \geq 3$

5. $\binom{n}{3} \equiv \frac{n!}{(n-3)!3!}$ for $n \geq 3$

2.2 Solutions

1. $\Delta n^4 = (n+1)^4 - n^4 = n^4 + 4n^3 + 6n^2 + 4n + 1 - n^4 = 4n^3 + 6n^2 + 4n + 1$
2. $\Delta e^n = e^{n+1} - e^n = e \cdot e^n - e^n = (e-1)e^n$
3. $\Delta \ln(n) = \ln(n+1) - \ln(n) = \ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right) \approx \frac{1}{n}$
4.

$$\begin{aligned}
\Delta n^{\underline{3}} &= (n+1)^{\underline{3}} - n^{\underline{3}} \\
&= \frac{(n+1)!}{(n+1-3)!} - \frac{n!}{(n-3)!} \\
&= \frac{(n+1)!}{(n-2)!} - \frac{(n-2)\cdot n!}{(n-2)!} \\
&= \frac{(n+1)\cdot n! - (n-2)\cdot n!}{(n-2)!} \\
&= \frac{(n+1-(n-2))\cdot n!}{(n-2)!} \\
&= 3\frac{n!}{(n-2)!} \\
&= 3n^{\underline{2}}
\end{aligned}$$

5.

$$\begin{aligned}
\Delta \binom{n}{3} &= \binom{n+1}{3} - \binom{n}{3} \\
&= \frac{(n+1)!}{(n+1-3)!3!} - \frac{n!}{(n-3)!3!} \\
&= \frac{1}{3!}\left[\frac{(n+1)!}{(n-2)!} - \frac{n!}{(n-3)!}\right] \\
&= \frac{1}{3!} \cdot \Delta n^{\underline{3}} \\
&= \frac{1}{3!} \cdot 3\frac{n!}{(n-2)!} \\
&= \frac{n!}{(n-2)!2!} \\
&= \binom{n}{2}
\end{aligned}$$

3 More on the Difference Operator

This is fun, isn't it? We now prove an important theorem about the difference operator.

Theorem 1. *Let a_n and b_n be sequences, and let c be any number. Then*

1. $\Delta(a_n + b_n) = \Delta a_n + \Delta b_n$, *and*
2. $\Delta(c \cdot a_n) = c \cdot \Delta a_n$.

Proof. We simply calculate

$$\begin{aligned}\Delta(a_n + b_n) &= (a_{n+1} + b_{n+1}) - (a_n + b_n) \\ &= (a_{n+1} - a_n) + (b_{n+1} - b_n) \\ &= \Delta a_n + \Delta b_n\end{aligned}$$

and

$$\begin{aligned}\Delta(c \cdot a_n) &= c \cdot a_{n+1} - c \cdot a_n \\ &= c \cdot (a_{n+1} - a_n) \\ &= c \cdot \Delta a_n\end{aligned}$$

□

In the language of mathematics, Theorem 1 tells us that the difference operator is a *linear* operator. This is a very convenient property for an operator to have, as will become evident in the following proposition.

Proposition 1 (The difference of a polynomial). *Let a_n be a polynomial in n of degree $k \geq 1$. Then Δa_n is a polynomial of degree $k - 1$.*

We will begin by proving a short lemma.

Lemma 1. $\Delta n^k = \sum_{i=0}^{k-1} \binom{k}{i} n^i$ *for all $k = 1, 2, 3, \ldots$.*

Proof. By the binomial theorem

$$\begin{aligned}(n+1)^k &= \sum_{i=0}^{k} \binom{k}{i} n^i \\ &= \binom{k}{0} + \binom{k}{1} n + \binom{k}{2} n^2 + \cdots + \binom{k}{k-1} n^{k-1} + \binom{k}{k} n^k.\end{aligned}$$

Notice $\binom{k}{k} = \frac{k!}{(k-k)!k!} = 1$ so

$$\begin{aligned}\Delta n^k &= (n+1)^k - n^k \\ &= \sum_{i=0}^{k} \binom{k}{i} n^i - n^k \\ &= \sum_{i=0}^{k-1} \binom{k}{i} n^i + \binom{k}{k} n^k - n^k \\ &= \sum_{i=0}^{k-1} \binom{k}{i} n^i\end{aligned}$$

□

Proof of Proposition 1. If a_n is a polynomial in n of degree k, then

$$a_n = \sum_{i=0}^{k} c_i n^i = c_0 + c_1 n + c_2 n^2 + \cdots + c_{k-1} n^{k-1} + c_k n^k$$

for some constants $c_0, c_1, \ldots, c_k$ where $c_k \neq 0$. Then by the linearity of the difference opeartor

$$\Delta a_n = \Delta \sum_{i=0}^{k} c_i n^i = \sum_{i=0}^{k} c_i \Delta n^i.$$

We just saw in Lemma 1 that Δn^i is a polynomial of degree $i-1$ for $i \geq 1$. For $i = 0$, $\Delta n^0 = \Delta 1 = 0$. Therefore, Δa_n is the sum of polynomials of degrees 0 through $k-1$. Thus Δa_n is a polynomial of degree $k-1$. □

Now it is your turn to compute the difference of a few general sequences. Use the linearity of the difference operator when possible.

3.1 Exercises

Compute the difference of the following sequences. Here $b > 0$, c is a constant and k is a natural number such that $1 \leq k \leq n$.

1. $c \cdot b^n$ (*exponential sequence*)

2. $c \cdot \log_b(n)$ (*logarithmic sequence*)

3. $n^{\underline{k}} = \frac{n!}{(n-k)!}$ (*falling factorial sequence*)

4. $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ (*binomial coefficient sequence*)

3.2 Solutions

1.

$$\begin{aligned}\Delta c \cdot b^n &= c\Delta b^n \\ &= c(b^{n+1} - b^n) \\ &= c(b \cdot b^n - b^n) \\ &= c(b-1)b^n\end{aligned}$$

2.

$$\begin{aligned}\Delta c \log_b(n) &= c\Delta \log_b(n) \\ &= c\left(\log_b(n+1) - \log_b(n)\right) \\ &= c\log_b\left(\frac{n+1}{n}\right) \\ &= c\log_b\left(1 + \frac{1}{n}\right)\end{aligned}$$

3.

$$\begin{aligned}\Delta n^{\underline{k}} &= (n+1)^{\underline{k}} - n^{\underline{k}} \\ &= \frac{(n+1)!}{(n+1-k)!} - \frac{n!}{(n-k)!} \\ &= \frac{(n+1)!}{(n+1-k)!} - \frac{(n+1-k)\cdot n!}{(n+1-k)!} \\ &= \frac{(n+1)\cdot n! - (n+1-k)\cdot n!}{(n+1-k)!} \\ &= \frac{(n+1-(n+1-k))\cdot n!}{(n+1-k)!} \\ &= \frac{k \cdot n!}{(n+1-k)!} \\ &= k\frac{n!}{(n-(k-1))!} \\ &= kn^{\underline{k-1}}\end{aligned}$$

4. This follows easily from the previous exercise since $\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{1}{k!}n^{\underline{k}}$:

$$\begin{aligned}
\Delta\binom{n}{k} &= \Delta\frac{1}{k!}n^{\underline{k}} \\
&= \frac{1}{k!}\Delta n^{\underline{k}} \\
&= \frac{k}{k!}n^{\underline{k-1}} \\
&= \frac{1}{(k-1)!}\frac{n!}{(n-(k-1))!} \\
&= \frac{n!}{(n-(k-1))!(k-1)!} \\
&= \binom{n}{k-1}
\end{aligned}$$

One could also see this by considering Pascal's triangle.

3.3 Take-Home Exercises

For the first three exercises, it is helpful to recall that

$$n^{\underline{k}} = \frac{n!}{(n-k)!} = n(n-1)(n-2)\cdots(n-k+2)(n-k+1)$$

That is, $n^{\underline{k}}$ is a polynomial in n of degree k.

1. Give the formula for a sequence a_n such that $\Delta a_n = n$.
   (hint: notice $n = n^{\underline{1}}$ and recall that $\Delta n^{\underline{2}} = 2n^{\underline{1}}$)

2. Give the formula for a sequence b_n such that $\Delta b_n = n^2$.
   (hint: notice $n^2 = n^{\underline{2}} + n^{\underline{1}}$)

3. Give the formula for a sequence c_n such that $\Delta c_n = n^3$.
(hint: write n^3 as a linear combination of $n^{\underline{3}}$, $n^{\underline{2}}$ and $n^{\underline{1}}$)

4. Give the formula for a sequence d_n such that $\Delta d_n = d_n$.
(hint: consider an exponential sequence)

3.4 Solutions to Take-Home Exercises

1. $a_n = \frac{1}{2}n^{\underline{2}} = \frac{n(n-1)}{2}$, for example
2. $b_n = \frac{1}{3}n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{n(n-1)(2n-1)}{6}$, for example
3. $c_n = \frac{1}{4}n^{\underline{4}} + n^{\underline{3}} + \frac{1}{2}n^{\underline{2}} = \frac{n^2(n-1)^2}{4}$, for example
4. $d_n = 2^n$, for example