MTH 370, Fall 2009
Solutions to Homework 6

Instructions: Do these calculations by hand (you may use a computer or calculator for simple arithmetic and function evaluations) and show your work.

1. Solve the following first-order ODEs using either separation of variables or integrating factors. Then determine the limit of the solution as $t \to \infty$.

   (a) $\frac{dx}{dt} = rx(1-x) \quad (r > 0)$

   (b) $\frac{dx}{dt} = -rx\ln(x) \quad (r > 0)$

   (c) $\frac{dx}{dt} = rx - e^{-t} \quad (r > 0)$

   Solutions:

   (a)

$$\begin{aligned}
\frac{dx}{dt} &= rx(1-x) \\
\frac{dx}{x(1-x)} &= r\,dt && \text{(separate variables)} \\
\ln\left|\frac{x}{1-x}\right| &= rt + c && \text{(integrate using partial fraction decomposition)} \\
\frac{x}{1-x} &= ce^{rt} && \text{(exponentiate and absorb sign in constant)} \\
x(t) &= \frac{c}{c + e^{-rt}} && \text{(solve for } x\text{)}
\end{aligned}$$

$$\lim_{t\to\infty} x(t) = \begin{cases} 1 & c > 0 \\ 0 & c = 0 \end{cases}$$

   (b)

$$\begin{aligned}
\frac{dx}{dt} &= -rx\ln(x) \\
\frac{dx}{x\ln(x)} &= -r\,dt && \text{(separate variables)} \\
\ln|\ln|x|| &= -rt + c && \text{(integrate using u-substitution)} \\
x(t) &= e^{ce^{-rt}} && \text{(exponentiate twice and absorb sign in constant)}
\end{aligned}$$

$$\lim_{t\to\infty} x(t) = \begin{cases} 1 & c \in (-\infty, \infty) \\ 0 & c = -\infty \end{cases}$$

(c)

$$\begin{aligned}
\frac{dx}{dt} &= rx - e^{-t} \\
\frac{dx}{dt} - rx &= -e^{-t} \\
\frac{d}{dt}(e^{-rt}x) &= -e^{-(r+1)t} \\
e^{-rt}x(t) &= \frac{e^{-(r+1)t}}{r+1} + c \\
x(t) &= \frac{e^{-t}}{r+1} + ce^{rt}
\end{aligned}$$

$$\lim_{t\to\infty} x(t) = \begin{cases} 1 & c > 0 \\ 0 & c = 0 \end{cases}$$

2. Find the equilibria of the following first-order ODEs and determine their stability. Then draw a phase-line diagram illustrating your findings.

   (a) $\frac{dx}{dt} = -rx(x-\alpha)(x-1) \quad (r > 0, \alpha \in (0,1))$

   (b) $\frac{dx}{dt} = -rx(\ln(x) - 1) \quad (r > 0)$

   Solutions:

   (a) $0 = f(x^*) = -rx^*(x^* - \alpha)(x^* - 1) \Rightarrow x^* = 0, \alpha, 1$
   $f'(x) = -r(3x^2 - 2(1+\alpha)x + \alpha)$
   $f'(0) = -r\alpha < 0 \Rightarrow$ stable
   $f'(\alpha) = r\alpha(1-\alpha) > 0 \Rightarrow$ unstable
   $f'(1) = -r(1-\alpha) < 0 \Rightarrow$ stable

   (b) $0 = f(x^*) = -rx^*(\ln(x^*) - 1) \Rightarrow x^* = 0, e$
   $f'(x) = -r\ln(x)$
   $f'(0) = \infty > 0 \Rightarrow$ unstable
   $f'(e) = -r < 0 \Rightarrow$ stable