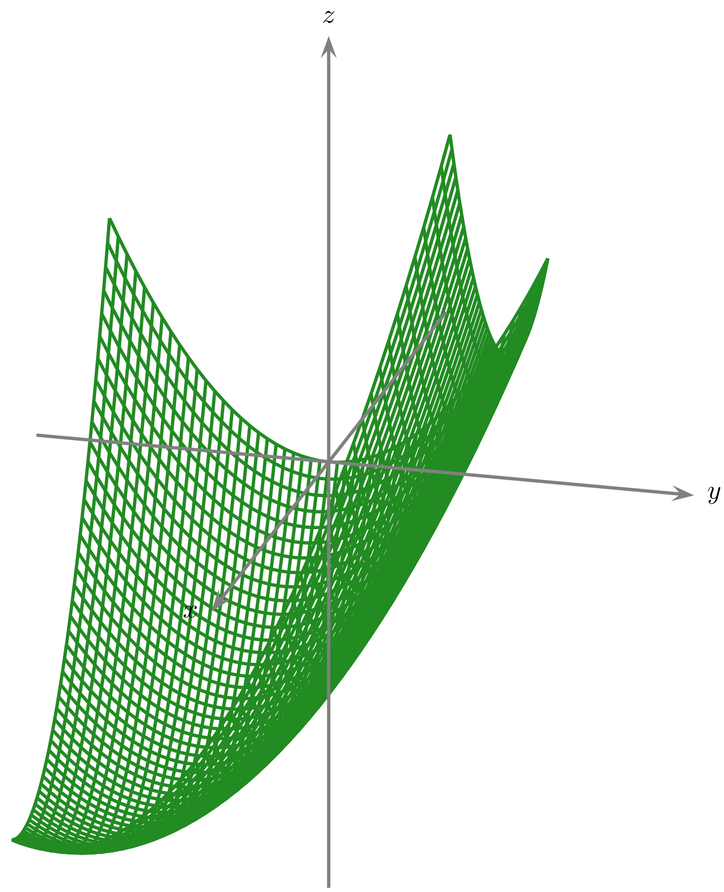
Let $z = f(x, y) = x^2 + xy + y^2 - 6x$ and let R be defined by $R: 0 \le x \le 5,\ -3 \le y \le 3$. Find the maximum and minimum values of f over the region R.



We must find all critical points in the interior of R and then check the boundary of R.

1. Find the critical points.

$$f_x = 2x + y - 6, \quad f_y = x + 2y$$

   Setting the partials equal to zero and solving the resulting (linear) system of equations, we see that $P = P(4, -2)$ is the only critical point and $P \in R$. See Figure 1.

2. Check the boundary. We will obviously need to check the four vertices of R. To see if there are other boundary points that should be included, we work with the function of one variable

$$f_j = f\Big|_{S_j}, \quad j = 1, 2, 3, 4.$$

   where S_1, S_2, S_3, S_4 are the four sides of the rectangular region R (see Figure 1).

   (a) $S_1:\ y = 3$. Now $f_2(x) = f(x, 3) = x^2 - 3x + 9,\ x \in S_1$ has a critical point at $x = 3/2$.

   (b) $S_2:\ x = 5$. Notice that
$$f_3(y) = f(5, y) = y^2 + 5y - 5,\ y \in S_2$$
   has a critical point at $y = -5/2$.

   (c) $S_3:\ y = -3$. Notice that
$$f_4(x) = f(x, -3) = x^2 - 9x + 9,\ x \in S_3$$
   has a critical point at $x = 9/2$.

   (d) $S_4:\ x = 0$. Then $f_1(y) = f(0, y) = y^2,\ y \in S_1$ has a critical point at $y = 0$.

   Putting these facts together we must compare the following values:

(x, y)	$f(x, y)$
$(0, 0)$	$f(0, 0) = 0$
$(0, 3)$	$f(0, 3) = 9$
$(5, 3)$	$f(5, 3) = 19$
$(5, -3)$	$f(5, -3) = -11$
$(0, -3)$	$f(0, -3) = 9$
$(3/2, 3)$	$f(3/2, 3) = 27/4$
$(5, -5/2)$	$f(5, -5/2) = -45/4$
$(9/2, -3)$	$f(9/2, -3) = -45/4$
$(4, -2)$	$f(4, -2) = -12$

So the absolute minimum is $f_{\min} = f(4, -2) = -12$ and the absolute maximum is $f_{\max} = f(5, 3) = 19$.

The sketch below identifies the nine domain values that we tested in this example. The function, $f(x, y)$ attains a global maximum at $(5, 3)$ (shown in blue) and a global minimum at $(4, -2)$ (shown in red).

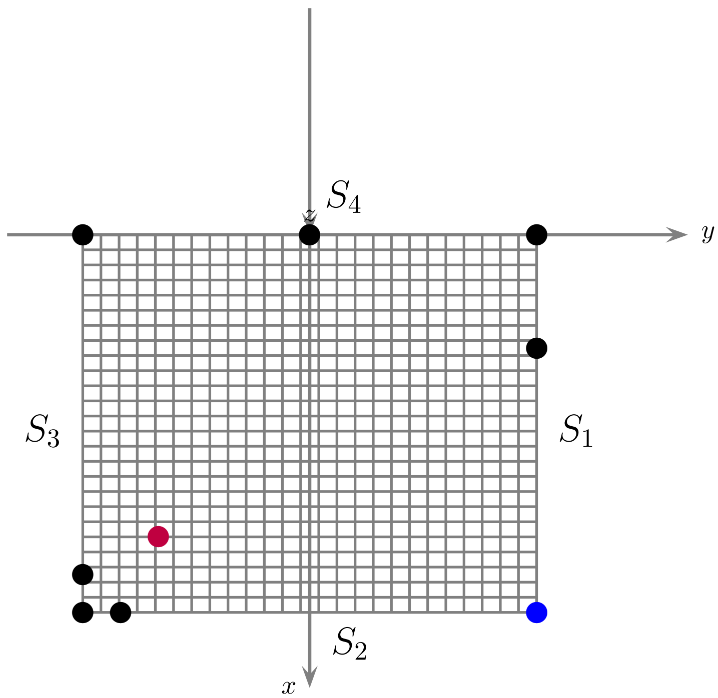


Figure 1: Test Points in the domain R for $f(x, y) = x^2 + xy + y^2 - 6x$.