Example. In section 14.3 we used partial derivatives to quickly find dy/dx for an equation that defined y implicitly as a function of x. One of our examples was similar to the equation

$$2x^2 - xy + y^2 = 8 \tag{1}$$

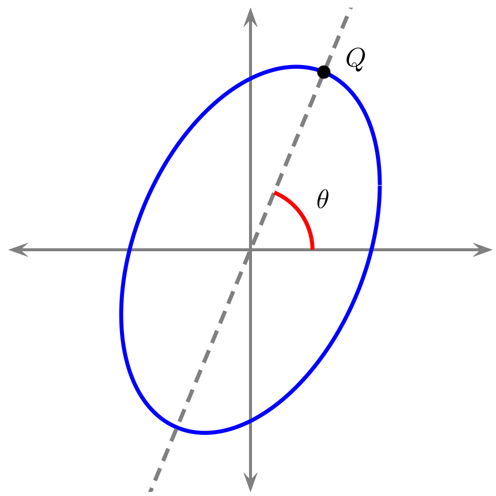

Figure 1: Level Curve: $g(x, y) = 8$

Now let $g(x, y) = 2x^2 - xy + y^2$. Then (1) is simply the level curve $g(x, y) = 8$. A sketch of this level curve is shown in Figure 1.

Notice that this is an ellipse with its major axis rotated counterclockwise by some angle θ. But what is θ, or equivalently, what are the coordinates of Q? It turns out that for an ellipse defined by the equation $Ax^2 + Bxy + Cy^2 = F$, we have

$$\tan 2\theta = \frac{B}{A - C}$$

There are at least two other methods from calculus that can be used to find θ. Can you describe them? Below we rewrite (1) in parametric form and give a few hints along the way.

Rewriting equation (1) we have

$$\begin{aligned} 8 &= y^2 - xy + \frac{x^2}{4} + \frac{7x^2}{4} \\ &= (y - x/2)^2 + \frac{x^2}{4} \end{aligned}$$

or

$$1 = \frac{(y - x/2)^2}{\left(\sqrt{8}\right)^2} + \frac{x^2}{\left(\sqrt{32/7}\right)^2}$$

Now we let

$$x = \sqrt{\frac{32}{7}} \cos t \quad \text{and} \quad y = \sqrt{8} \sin t + \frac{x}{2} = \sqrt{8} \sin t + \sqrt{\frac{32}{7}} \cos t$$

With the help of the addition formula for sine, we can rewrite the second parametric equation in a more compact form. That is,

$$x = \sqrt{\frac{32}{7}}\cos t$$

$$y = \sqrt{\frac{64}{7}}\sin(t+\alpha)$$

where $\alpha = \arcsin\frac{1}{\sqrt{8}}$. You can see a parametric plot of these equations here.

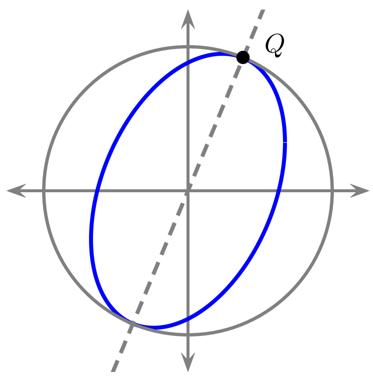

Figure 2: Level Curve with Circle

Returning to the task of finding the coordinates of Q. The first hint is shown in Figure 2.

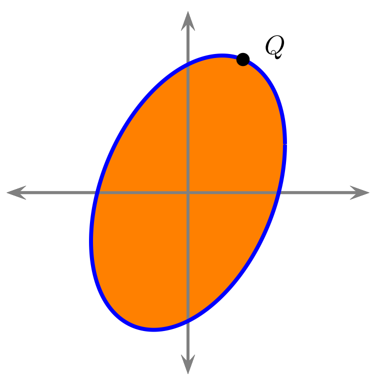

Figure 3: Domain of $f(x, y)$.

For a different approach, can you define a function $f(x, y)$ on the ellipse and its interior (see Figure 3) that attains its maximum value at Q.