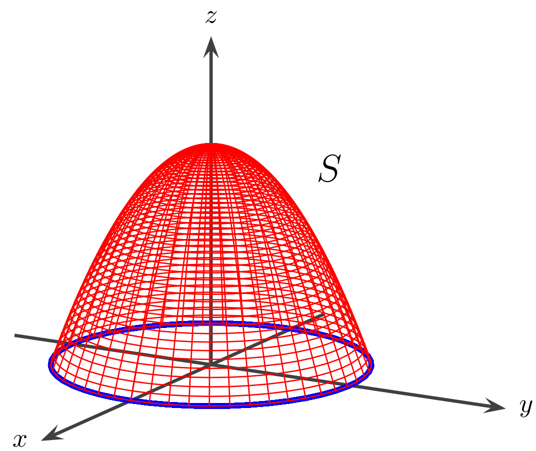

Figure 1: A parabolic cap

Example 1. Let $\mathbf{G} = 3yz\,\mathbf{i} + x\,\mathbf{j} + xy\,\mathbf{k}$. Let S (see Fig. 1) be the level surface of $g(x, y, z) = x^2 + y^2 + z = 9, \;\; z \geq 0$, oriented so that the vector normal has a positive $\mathbf{k}$ component. Evaluate the surface integral below using 3 different methods as we did in Example 16.8.1.

(1) $$\iint_S \nabla \times \mathbf{G} \cdot \mathbf{n}\, dS$$

Remark. Notice that the vector field $\mathbf{G}$ is different than the vector field $\mathbf{F} = 2y\,\mathbf{i} - 3x\,\mathbf{j} - z^2\,\mathbf{k}$ given in Example 16.8.1. In particular,

(2) $$\begin{aligned} \nabla \times \mathbf{G} &= x\,\mathbf{i} + 2y\,\mathbf{j} + (1 - 3z)\,\mathbf{k} \\ &\neq -5\,\mathbf{k} = \nabla \times \mathbf{F} \end{aligned}$$

(a) **Direct Computation**

As we saw in Example 16.8.1, the vector equation for S is given by

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + (9 - x^2 - y^2)\,\mathbf{k},\ \ (x,y) \in R$$

where $R = \{(x,y)\,|\,x^2 + y^2 \leq 9\}$. Now

$$\mathbf{r}_x = \mathbf{i} - 2x\,\mathbf{k} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$$

so that

(3) $$\mathbf{r}_x \times \mathbf{r}_y = 2x\,\mathbf{i} + 2y\,\mathbf{j} + \mathbf{k}$$

Thus

$$\nabla \times \mathbf{G}(\mathbf{r}(x,y)) = x\,\mathbf{i} + 2y\,\mathbf{j} + (1 - 3(9 - x^2 - y^2))\,\mathbf{k}$$

Now combine (2) and (3) to obtain

$$\nabla \times \mathbf{G}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = 5x^2 + 7y^2 - 26$$

It follows that

$$\begin{aligned}
\iint_S \nabla \times \mathbf{G} \cdot \mathbf{n}\,dS &= \iint_R \nabla \times \mathbf{G}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\,dA \\
&= \iint_R 5x^2 + 7y^2 - 26\,dA \\
&= 5\iint_R x^2\,dA + 7\iint_R y^2\,dA - 26\iint_R dA \\
&= 5 \times \frac{81\pi}{4} + 7 \times \frac{81\pi}{4} - 26 \times 9\pi \\
&= 9\pi
\end{aligned}$$

(b) **Using Stokes' Theorem**

Notice that the boundary of S lives in the xy-plane. We first compute the (counterclockwise) **circulation** around the closed curve ∂S which has the vector equation

$$\partial S: \quad \mathbf{r}(t) = 3\cos t\ \mathbf{i} + 3\sin t\ \mathbf{j} + 0\,\mathbf{k}, \quad 0 \le t \le 2\pi$$

Thus

$$\begin{aligned} d\mathbf{r}(t) &= -3\sin t\, dt\ \mathbf{i} + 3\cos t\, dt\ \mathbf{j} \\ \mathbf{F} &= 3yz\,\mathbf{i} + x\,\mathbf{j} + xy\,\mathbf{k} \\ \mathbf{F}(\mathbf{r}(t)) &= 0\,\mathbf{i} + 3\cos t\ \mathbf{j} + 9\sin t\ \cos t\ \mathbf{k} \end{aligned}$$

so that

$$\mathbf{F}(\mathbf{r}(t)) \cdot d\mathbf{r} = 9\cos^2 t\, dt$$

So by Stokes' Theorem

$$\begin{aligned} \iint_S \nabla \times \mathbf{G} \cdot \mathbf{n}\, dS &= \oint_C \mathbf{F} \cdot d\mathbf{r} \\ &= 9 \int_0^{2\pi} \cos^2 t\, dt \\ &= 9\pi \end{aligned}$$

as we saw in part (a).

Notice that the circulation integral calculation above was a bit easier the surface integral calculation in part (a).

(c) **Exploiting Green's Theorem**

As we observed above, ∂S happens to lie in the xy-plane. Now let R be as indicated in part (a). Then by Stokes' Theorem and (the tangential form of) Green's Theorem, we have

$$\begin{aligned} \iint_S \nabla \times \mathbf{G} \cdot \mathbf{n}\, dS &= \oint_{\partial S} \mathbf{G} \cdot d\mathbf{r} \\ &= \iint_R \nabla \times \mathbf{G} \cdot \mathbf{k}\, dA \\ &= \iint_R (1 - 3z)\, dA \\ &= \iint_R (1 - 0)\, dA, \quad \text{(since } z = 0 \text{ in the } xy\text{-plane)} \\ &= 9\pi \end{aligned}$$