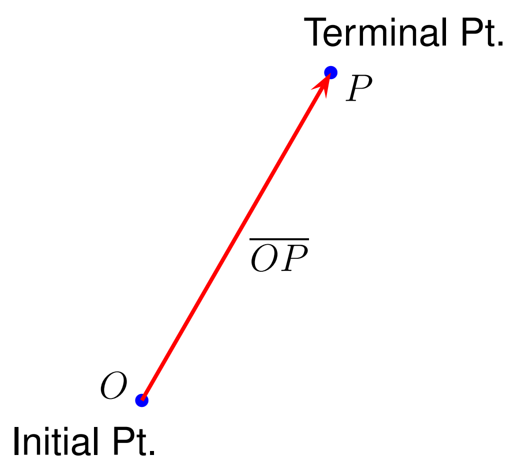
12.2 Vectors

Some things that we measure are determined simply by their magnitude, e.g., length, time, mass, etc. Others require more information. For example, to describe velocity one needs the “speed” and direction.

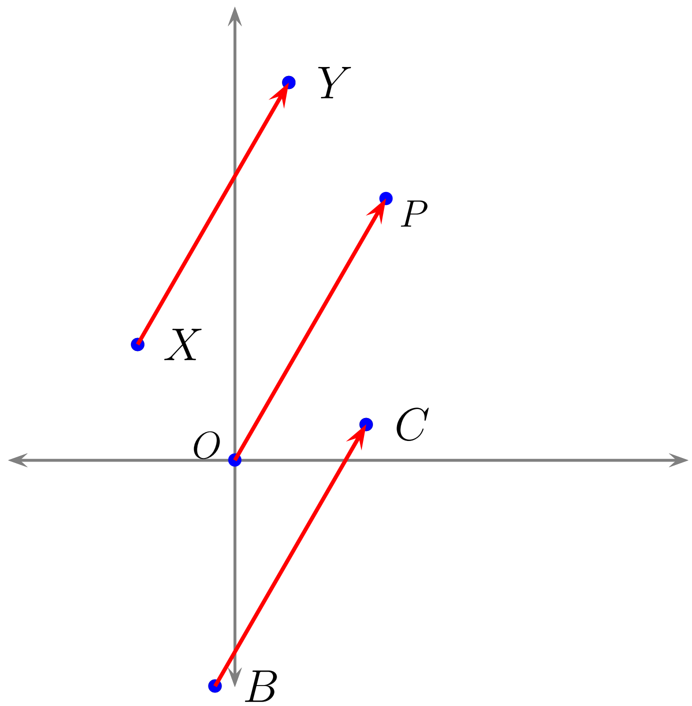
Component Form

In mathematics (and physics...) such quantities are called **vectors** and are represented by a *directed line segment*.



Definition. A **vector** in the plane (or in space) is a directed line segment. The directed line segment $\overline{OP}$ has initial and terminal points O and P respectively and its **length** (also called magnitude) is denoted by $\left|\overline{OP}\right|$. Two vectors are **equal** if the have the same length and direction.

It follows from this last statement that the three directed line segments in the figure below represent the same vectors since they have the same length and direction.



Now let $\vec{v} = \overline{XY}$ (e.g., as shown in the above sketch). Each of the other directed line segments in the above sketch is also a representative of $\vec{v}$. The directed line segment whose initial point is located at the origin is the representative of $\vec{v}$ in *standard position* and usually our choice candidate to "represent" $\vec{v}$.

With this agreement we can now represent $\vec{v}$ in **component form** by simply indicating the coordinates of the terminal point (the initial point assumed to be the origin).

Definition. Vectors - Component Form

Suppose that $\vec{v}$ is the vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2). Then the **component form** of $\vec{v}$

$$\vec{v} = \langle v_1, v_2 \rangle$$

Of course, in three dimensions we have

$$\vec{v} = \langle v_1, v_2, v_3 \rangle$$

Remark.

i. If $\vec{v} = \langle v_1, v_2, v_3 \rangle$ then the real numbers v_1, v_2, v_3 are called the **components** of $\vec{v}$.

ii. Let $P = P(x_1, y_1, z_1)$ and $Q = Q(x_2, y_2, z_2)$ and let $\vec{v} = \overline{PQ}$. Then the component form of $\vec{v}$ is

$$\vec{v} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle$$

Example 1. Let $P = (2, 3, -4)$ and $Q = (1, -1, 2)$. Find the component form of $\vec{v} = \overline{PQ}$.

The **magnitude** or **length** of the vector $\vec{v}$ is the length of any of its equivalent directed line segments and is denoted $|\vec{v}|$ or $\|\vec{v}\|$. Notice that if $\vec{v} = \langle v_1, v_2, v_3\rangle = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1\rangle$ then

$$\|\vec{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2} \tag{1}$$

$$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \tag{2}$$

Example 2. Find the magnitude of the vector from the previous example. Recall that $\vec{v} = \langle -1, -4, 6\rangle$ so that

$$\begin{aligned} \|\vec{v}\| &= \sqrt{(-1)^2 + (-4)^2 + (6)^2} \\ &= \sqrt{53} \end{aligned}$$

Vector Arithmetic

Definition. Let $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$ and let $k \in \mathbb{R}$. (The real number k is called a **scalar** for reasons that will become clear below.). Then we define two new vectors $\vec{u} + \vec{v}$ and $k\vec{u}$ as follows.

> **Vector Addition:** $\vec{u} + \vec{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$ and
>
> **Scalar Multiplication:** $k\,\vec{u} = \langle ku_1, ku_2, ku_3 \rangle$.

Remark. It is important to emphasize that these operations yield vector quantities. In other words, the collection of vectors are *closed* under vector addition and scalar multiplication. Also, vector arithmetic has geometric interpretations.

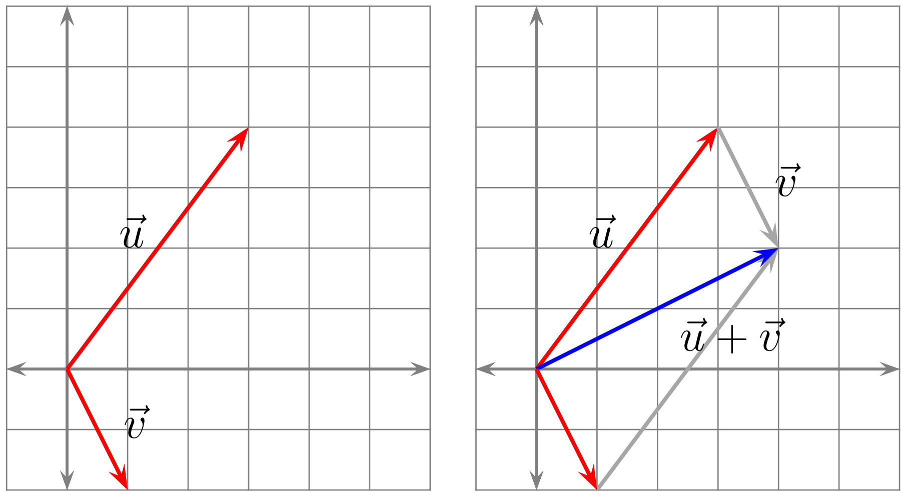
i. Vector addition can be visualized geometrically using the **parallelogram law**.

ii. The vector $k\vec{u}$ is a "scaled" version of $\vec{u}$.

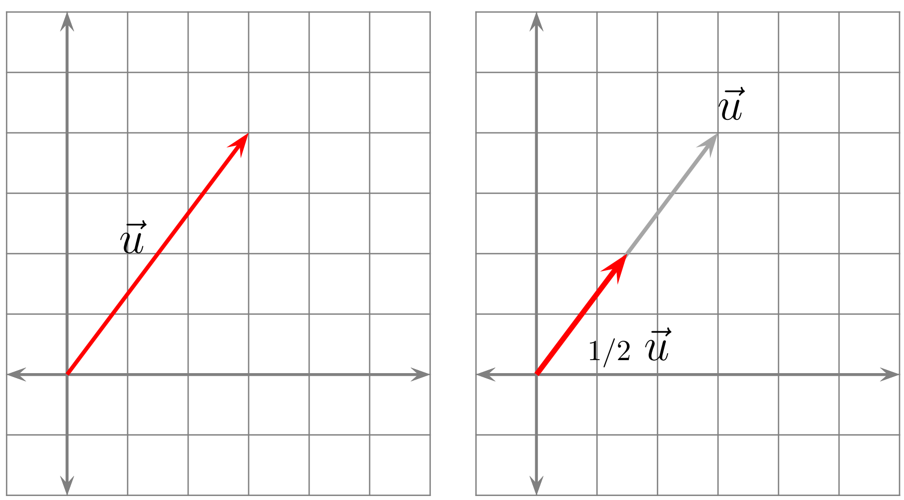
The example below illustrates these ideas.

Example 3. Let $\vec{u} = \langle 3, 4\rangle$ and $\vec{v} = \langle 1, -2\rangle$. Find $\vec{u} + \vec{v}$ and $\dfrac{1}{2}\vec{u}$.

1. $\vec{u} + \vec{v} = \langle 3 + 1, 4 + (-2)\rangle = \langle 4, 2\rangle$.



2. $\dfrac{1}{2}\vec{u} = \langle 3/2, 2\rangle$.



Example 4. Let $\vec{u} = \langle 3, 4 \rangle$ and $\vec{v} = \langle 1, -2 \rangle$. Find $2\vec{u} - \vec{v}$ and $\|3\vec{v}\|$.

It is important to mention the zero vector $\mathbf{0} = \langle 0, 0 \rangle$ or $\mathbf{0} = \langle 0, 0, 0 \rangle$ as the only vector of *zero* length and any direction.

Proposition 1.

$$\|k\vec{u}\| = |k|\|\vec{u}\|, \quad k \in \mathbb{R} \tag{3}$$

Proof. Write $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and let $k \in \mathbb{R}$.

$$\begin{aligned}
\|k\vec{u}\|^2 &= \|\langle ku_1, ku_2, ku_3 \rangle\|^2 \\
&= (ku_1)^2 + (ku_2)^2 + (ku_3)^2 \\
&= k^2 \left(u_1^2 + u_2^2 + u_3^2\right) \\
&= k^2 \|\vec{u}\|^2
\end{aligned}$$

Now the result follows by taking square roots. (Why?)

We list several other important properties of vectors.

Proposition 2. Vector Properties

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors and $a, b \in \mathbb{R}$. Then

1. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. $\vec{u} + \mathbf{0} = \vec{u}$
4. $\vec{u} + (-\vec{u}) = \mathbf{0}$
5. $0\vec{u} = \mathbf{0}$
6. $1\vec{u} = \vec{u}$
7. $a\,(b\vec{u}) = (ab)\,\vec{u}$
8. $a\,(\vec{u} + \vec{v}) = a\vec{u} + a\vec{v}$
9. $(a + b)\,\vec{u} = a\vec{u} + b\vec{u}$

Proof. Exercise.

Remark. Property 4 should be reworded. It should say that for each vector $\vec{u}$ there is a **unique** vector $\vec{v}$ such that $\vec{u} + \vec{v} = \mathbf{0}$. The vector $\vec{v}$ is usually denoted $-\vec{u}$. Please look these over very carefully.

Proposition 3. Vector Properties (cont.)

10. $-1\vec{u} = -\vec{u}$

Proof. We need to show that $-1\vec{u}$ is the additive inverse of $\vec{u}$. We could prove these by appealing to the component definitions of scalar multiplication. Instead we try another approach.

Identify the vector properties from Proposition 2 that are used in the proof below.

$$\begin{aligned} \mathbf{0} &= 0\vec{u} \\ &= (1 + -1)\vec{u} \\ &= 1\vec{u} + -1\vec{u} \\ &= \vec{u} + -1\vec{u} \end{aligned}$$

and the result follows. Why?

Unit Vectors

A vector of length 1 is called a **unit vector**. It is convenient to introduce the following unit vectors (called the **standard unit vectors**).

$$\vec{i} = \mathbf{i} = \langle 1, 0, 0 \rangle$$
$$\vec{j} = \mathbf{j} = \langle 0, 1, 0 \rangle$$
$$\vec{k} = \mathbf{k} = \langle 0, 0, 1 \rangle$$

Now if $\vec{u} = \langle u_1, u_2, u_3 \rangle$ then

$$\begin{aligned}
\vec{u} &= \langle u_1, u_2, u_3 \rangle \\
&= \langle u_1, 0, 0 \rangle + \langle 0, u_2, 0 \rangle + \langle 0, 0, u_3 \rangle \\
&= u_1 \langle 1, 0, 0 \rangle + u_2 \langle 0, 1, 0 \rangle + u_3 \langle 0, 0, 1 \rangle \\
&= u_1 \,\mathbf{i} + u_2 \,\mathbf{j} + u_3 \,\mathbf{k}
\end{aligned}$$

Remark. We often refer to $u_1 \,\mathbf{i} + u_2 \,\mathbf{j} + u_3 \,\mathbf{k}$ as a linear combination of the vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$. Also, $u_1, u_2,$ and u_3 are called, resp., the $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ components of $\vec{u}$ At first glance this notation may seem tedious but it does have some advantages (which will become clearer later in the course).

Now suppose that $\vec{u} \neq \mathbf{0}$ then $\|\vec{u}\| \neq 0$ and

$$\begin{aligned} 1 &= \frac{1}{\|\vec{u}\|}\,\|\vec{u}\| \\ &= \|\frac{1}{\|\vec{u}\|}\,\vec{u}\|, \quad \text{(by Proposition 1)} \\ &= \|\,\underbrace{\frac{1}{\|\vec{u}\|}}_{\text{scalar}}\;\underbrace{\vec{u}}_{\text{vector}}\,\| \end{aligned}$$

So the vector $\dfrac{1}{\|\vec{u}\|}\,\vec{u}$ is a unit vector in the direction of $\vec{u}$ and is called **the direction of** $\vec{u}$. And every non-zero vector can be written as

(4)
$$\vec{u} = \underbrace{\|\vec{u}\|}_{\text{length}} \cdot \underbrace{\frac{1}{\|\vec{u}\|}\vec{u}}_{\text{direction}}$$

Example 5. Let $\vec{u} = 3\,\mathbf{i} - 2\,\mathbf{j}$. Decompose $\vec{u}$ as shown in equation (4). In other words, rewrite $\vec{u}$ as a length $\times$ direction vector.