## 13.1 Vector Functions and Space Curves

If a particle is moving through space during a given time interval, say $t \in I = [t_0,\, t_1]$, then the position of the particle at time $t \in I$ is

$$\begin{aligned} P &= P(x, y, z) \\ &= P(f(t), g(t), h(t)) \end{aligned}$$

That is,

(1) $$x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I$$

Then for each $t \in I$, the points make up the **curve** in space called the particle's **path**. The equations in (1) **parameterize** the curve.

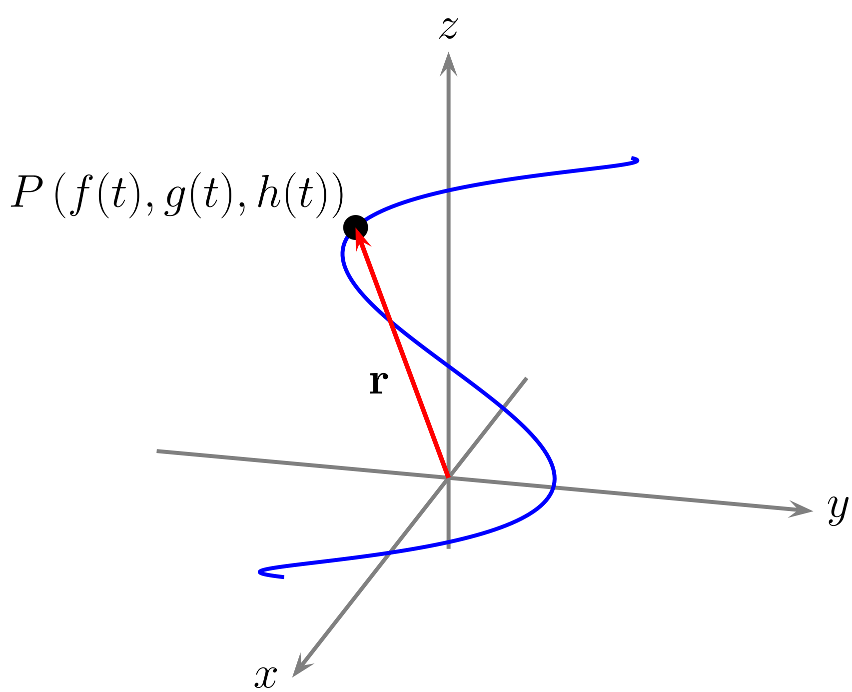
The vector

(2) $$\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$$

is a **position** vector of the particle at time $t$. And the functions $f,\ g,$ and $h$ are the **component functions** of $\mathbf{r}$.

We often refer to the function in (2) as (an example of) a **vector-valued** function. On the other hand, the component functions are examples of **scalar-valued** functions.

## Example 1. A Space Curve




We often like to think of $\mathbf{r}(t)$ as the *position vector* of a particle $P$, traveling along the curve, at time $t$.

**Example 2.** At which point(s) does the space curve $\mathbf{r}(t) = 2t\,\mathbf{i} - (t + t^2)\,\mathbf{k}$ intersect the paraboloid $z = x^2 + 4y^2$?

**Solution:**

It might be easier to attack this problem by rewriting the space curve in parametric form. That is,

(3) $$x(t) = 2t,\ y(t) = 0,\ z(t) = -t - t^2$$

It is then a bit easier to see that the curve will intersect the surface precisely when $P = P(x(t), y(t), z(t))$ satisfies the equation $z = x^2 + 4y^2$. That is, when

$$z(t) = x(t)^2 + 4y(t)^2$$

So we must solve

$$-t - t^2 = 4t^2 + 0$$

Rearranging, we obtain

$$0 = t(5t + 1)$$

It follows that the curve intersects the surface at $t = 0, -1/5$. From (3) we conclude that points of intersection are $(0, 0, 0)$ and $(-2/5, 0, 4/25)$.

**Example 3.** Find the parametric equations for the curve that represents the intersection of the surfaces below.

$$1 = \frac{x^2}{4} + \frac{y^2}{9} \tag{4}$$

$$z = 3 - 2x \tag{5}$$

**Solution:**

This one's a bit tricky if you forget to parameterize the elliptic cylinder. A (more or less) standard parameterization of (4) is given by

$$\begin{aligned} x(t) &= 2\cos t \\ y(t) &= 3\sin t \end{aligned} \tag{6}$$

Now (5) and (6) imply that

$$\begin{aligned} z(t) &= 3 - 2x(t) \\ &= 3 - 4\cos t \end{aligned}$$

See the blue curve in Figure 1 below.

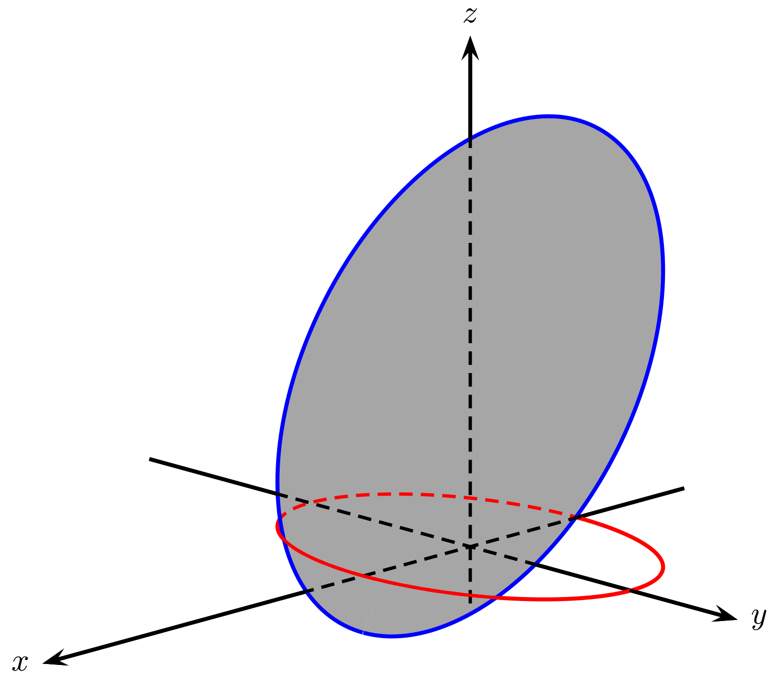



Figure 1: Intersection of Two Surfaces

The above figure requires some explanation. The blue curve is the intersection of the two surfaces. The red curve is the intersection of the elliptical cylinder $36 = 9x^2 + 4y^2$ with the $xy$-plane. The tilted gray ellipse includes all the points in the plane $z = 3 - 2x$ such that $9x^2 + 4y^2 < 1$. The paraboloid is not shown.

# Limits and Continuity

We define the limit of a vector-valued function component-wise. That is

**Definition. The Limit of a Vector-Valued Function**

Let $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ be a vector function and $\mathbf{L}$ be a vector. We say that $\mathbf{r}$ has a limit $\mathbf{L}$ as $t$ approaches $t_0$ if for every $\epsilon > 0$ there is a $\delta = \delta(\epsilon) > 0$ such that

(7) $$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \text{ whenever } 0 < |t - t_0| < \delta$$

In this case we write

(8) $$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

Fortunately, we have the following

**Proposition 1.** Let $\mathbf{L} = L_1\,\mathbf{i} + L_2\,\mathbf{j} + L_3\,\mathbf{k}$. Then (8) holds whenever

$$\lim_{t \to t_0} f(t) = L_1, \quad \lim_{t \to t_0} g(t) = L_2, \quad \lim_{t \to t_0} h(t) = L_3$$

In this case, we write

(9) $$\lim_{t \to t_0} \mathbf{r}(t) = \left(\lim_{t \to t_0} f(t)\right)\,\mathbf{i} + \left(\lim_{t \to t_0} g(t)\right)\,\mathbf{j} + \left(\lim_{t \to t_0} h(t)\right)\,\mathbf{k}$$

Finally, we have

**Definition. Continuity at a Point**

A vector-valued function $\mathbf{r}(t)$ is **continuous** at $t = t_0$ if

(10)
$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$$

The function is called continuous if (10) at every point in its domain.

*Remark.* By Proposition 1, a vector-valued function is continuous at $t = t_0$ if and only if each of its component functions are.

**Example 4. Limits and Continuity of a Space Curve**

Let $\mathbf{r}(t) = \cos t\,\mathbf{i} - 3t^2\,\mathbf{j} + \dfrac{1}{\sin t}\,\mathbf{k}$.

Then $\mathbf{r}(t)$ is continuous everywhere in its domain since each of its component functions are. What is its domain?