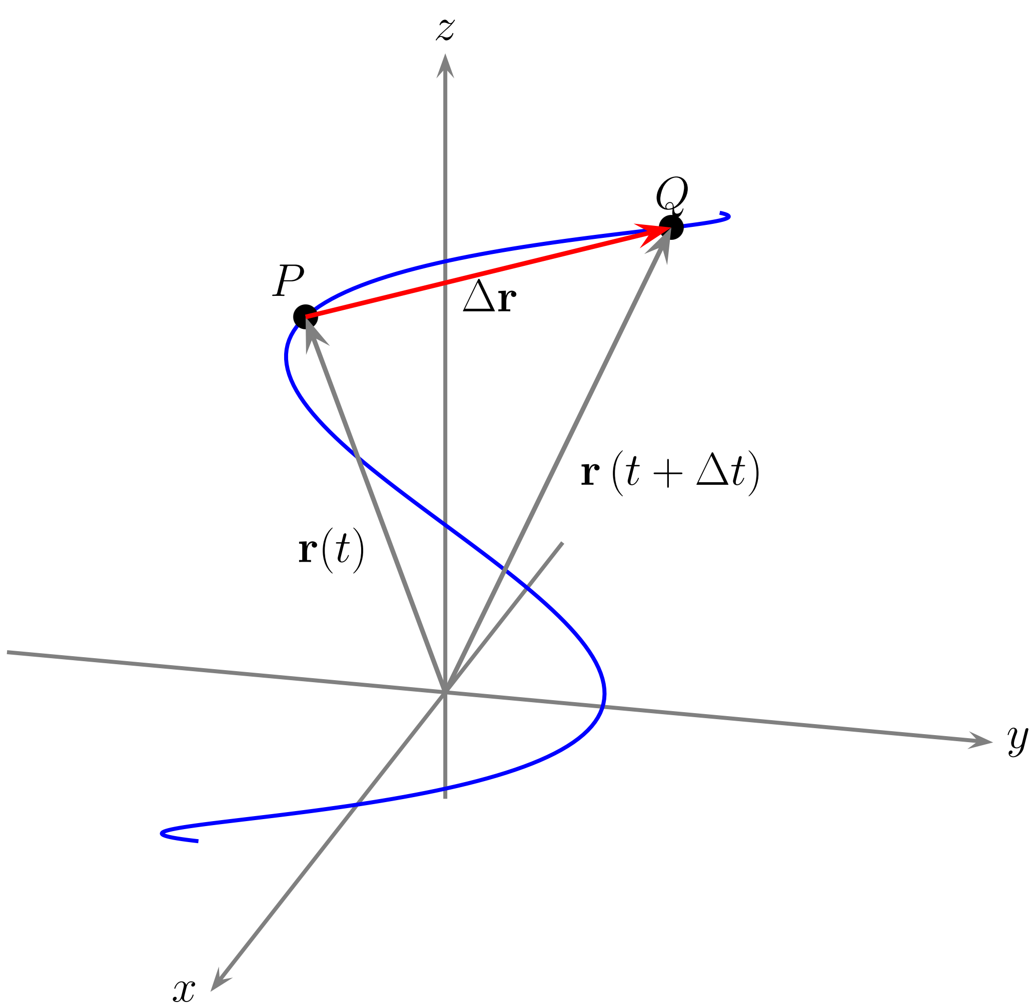
13.2 Derivatives and Integrals of Vector Functions

Derivatives and Motion

Suppose that $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is the position of a particle moving along a space curve (see sketch). Also, suppose that $f, g,$ and h are differentiable functions of t. Now the change in the position vector from time t to time $t + \Delta t$ is

(1) $$\Delta\mathbf{r} = \mathbf{r}\left(t + \Delta t\right) - \mathbf{r}(t)$$



Now if we rewrite (1) component-wise we get

$$\Delta\mathbf{r} \;=\; [f\,(t+\Delta t) - f(t)]\,\mathbf{i} \;\; + [g\,(t+\Delta t) - g(t)]\,\mathbf{j} + [h\,(t+\Delta t) - h(t)]\,\mathbf{k}$$

It follows that

$$\begin{aligned}
\lim_{\Delta t\to 0}\frac{\Delta\mathbf{r}}{\Delta t} &= \left[\lim_{\Delta t\to 0}\frac{f\,(t+\Delta t)-f(t)}{\Delta t}\right]\mathbf{i} \\
&\quad + \left[\lim_{\Delta t\to 0}\frac{g\,(t+\Delta t)-g(t)}{\Delta t}\right]\mathbf{j} + \left[\lim_{\Delta t\to 0}\frac{h\,(t+\Delta t)-h(t)}{\Delta t}\right]\mathbf{k} \\
&= \left(\frac{df}{dt}\right)\mathbf{i} + \left(\frac{dg}{dt}\right)\mathbf{j} + \left(\frac{dh}{dt}\right)\mathbf{k} \\
&= f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}
\end{aligned}$$

This leads to the following

Definition. The Derivative of a Vector-Valued Function

The vector-valued function $\mathbf{r}(t) = f(t)\,\mathbf{i} + g(t)\,\mathbf{j} + h(t)\,\mathbf{k}$ is **differentiable** at t provided each of the component functions is differentiable at t. In this case we have

(2) $$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = f'(t)\,\mathbf{i} + g'(t)\,\mathbf{j} + h'(t)\,\mathbf{k}$$

We make the usual observations regarding differentiability at a point versus differentiability at every point in the domain of the given function...

The curve traced by a vector-valued function $\mathbf{r}$ is called **smooth** if the derivative $d\mathbf{r}/dt$ is continuous and never $\mathbf{0}$. In other words, the component functions have continuous first derivatives that are *never* simultaneously 0.

Definition. Velocity, Direction, Speed, Acceleration

If $\mathbf{r}$ is the position vector of a particle moving along a *smooth* curve in space, then we have the following definitions.

1. **Velocity** is given by: $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \mathbf{r}'(t)$

2. The particle's **speed** is given by: Speed $= |\mathbf{v}|$

3. The **acceleration** is given by: $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2\mathbf{r}}{dt} = \mathbf{v}'(t)$

4. The **unit tangent vector** is the direction of motion at time t. That is,

$$\mathbf{T}(t) = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Proposition 1. Derivative Rules for Vector-Valued Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions of t, $\mathbf{C}$ be a constant vector, c a scalar, and f any differentiable scalar valued function. Then

1. $\dfrac{d}{dt}\mathbf{C} = \mathbf{0}$

2. $\dfrac{d}{dt}\left[c\mathbf{u}(t)\right] = c\mathbf{u}'(t)$

3. $\dfrac{d}{dt}\left[f(t)\mathbf{u}(t)\right] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

4. $\dfrac{d}{dt}\left[\mathbf{u}(t) \pm \mathbf{v}(t)\right] = \mathbf{u}'(t) \pm \mathbf{v}'(t)$

5. $\dfrac{d}{dt}\left[\mathbf{u}(t) \cdot \mathbf{v}(t)\right] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$

6. $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

7. $\dfrac{d}{dt}\left[\mathbf{u}\left(f(t)\right)\right] = \mathbf{u}'\left(f(t)\right) f'(t)$