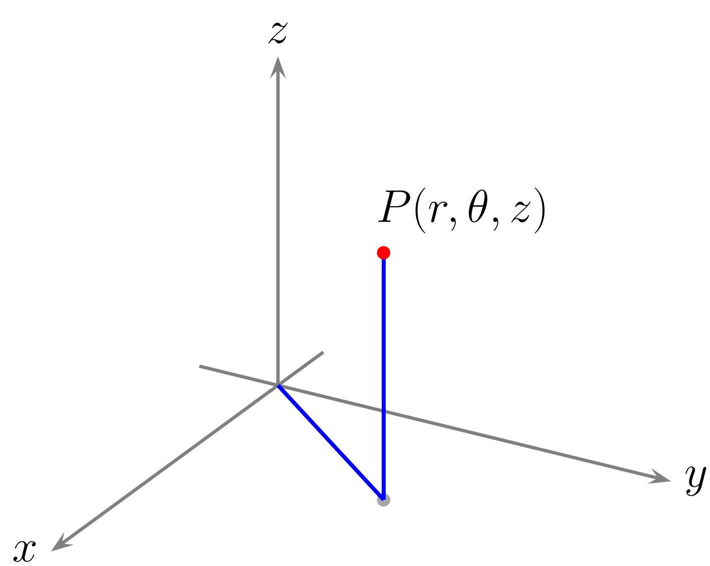
14.0 Cylindrical and Spherical Coordinates

Cylindrical Coordinates

Definition. **Cylindrical coordinates** represent a point P in space by the ordered triple (r, θ, z) where

a. r and θ are the polar coordinates for the vertical projection of P onto the xy-plane.

b. z is the rectangular vertical coordinate of P.



The following equations relate rectangular coordinates (x, y, z) to cylindrical coordinates (r, θ, z).

$$x = r\cos\theta, \quad y = r\sin\theta, \quad z = z$$

$$\big(\text{Also, } r^2 = x^2 + y^2 \ \text{ and } \ \tan\theta = y/x\big)$$

Remark. One must exercise care when using the second set of equations.

Example 1. Constant-Coordinate Equations

Describe the objects generated by the constant equations:

$$r = r_0$$

$$\theta = \theta_0$$

$$z = z_0$$

Example 2.

a. Convert the rectangular coordinates $(1, -2, 3)$ to cylindrical coordinates.

b. Convert the cylindrical coordinates $(3, 2\pi/3, -1)$ to rectangular coordinates.

Example 3. Describe the surface whose equation is given below. Convert the equation to rectangular or cylindrical coordinates, as appropriate.

a. Cylindrical Equation: $r = 3$

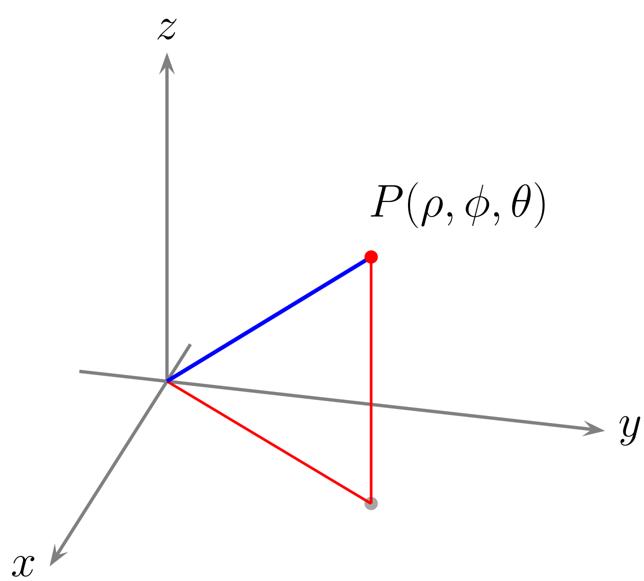
b. Cylindrical Equation: $z = 2r$

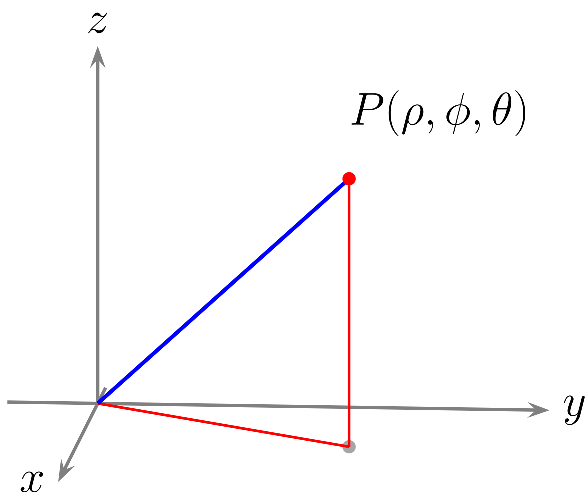
c. Rectangular Equation: $z = x^2 - y^2$

Spherical Coordinates

Definition. **Spherical coordinates** represent a point P in space by the ordered triple (ρ, ϕ, θ) where

a. ρ is the **distance** from P to the origin. So by definition $\rho \geq 0$.
b. ϕ is the angle that $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
c. θ is the angle as defined in the *cylindrical coordinate* system earlier today.





The following equations relate spherical coordinates to rectangular and cylindrical coordinates.

$$r = \rho \sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$

$$z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Example 4. Constant-Coordinates Equations

Describe the objects generated by the constant equations:

$$\rho = \rho_0$$

$$\phi = \phi_0$$

$$\theta = \theta_0$$

Example 5. Convert the rectangular coordinates $(0, 1, \sqrt{3})$ to spherical coordinates.

Example 6. Write the equations below in spherical or rectangular coordinates, as appropriate.

a. $z^2 = x^2 + y^2$

b. $\rho = 1 + \cos\phi$

Example 7.

Consider the spherical equation $\phi = \dfrac{\pi}{3}$. Find the equivalent cylindrical and rectangular equations.

1. **First Attempt:**

   (a) **Cylindrical Coordinate Equation:** We've already looked at the cross-sections $z = \text{const}\ (\geq 0)$. Notice that the if $y = 0$ we must have the equation $\tan\phi = x/z$. Thus

   $$\begin{aligned} &\frac{x}{z} = \tan\phi = \sqrt{3} \\ &\Longrightarrow x = \sqrt{3}\,z \\ &\Longrightarrow x^2 = 3z^2 \\ &\Longrightarrow r^2 = 3z^2 \quad \text{(Why?)} \end{aligned}$$

   (b) **Rectangular Coordinate Equation:** The last equation implies

   $$x^2 + y^2 = 3z^2$$

2. **Alternate Approach:**

$$
\begin{aligned}
\phi = \frac{\pi}{3} &\Longrightarrow \tan\phi = \sqrt{3} \\
&\Longrightarrow \frac{r}{z} = \sqrt{3} \\
&\Longrightarrow r^2 = 3z^2 \\
&\Longrightarrow \ldots
\end{aligned}
$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\ \cos\theta$	$x = \rho\ \sin\phi\ \cos\theta$	$r = \rho\ \sin\phi$
$y = r\ \sin\theta$	$y = \rho\ \sin\phi\ \sin\theta$	$z = \rho\ \cos\phi$
$z = z$	$z = \rho\ \cos\phi$	$\theta = \theta$