14.4 Tangent Planes and Linear Approximations

The Chain Rule

Theorem 1. Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y and z are differentiable functions of t, then w is a differentiable function of t and

(1) $$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Now let

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

and as usual, let

$$\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

Then (1) can be restated as

(2) $$\frac{dw}{dt} = \nabla f \cdot \frac{d\mathbf{r}}{dt}$$

Remark. ∇f is called the **gradient of** f. We will prove this theorem in section 14.5 and we will say more about the gradient in section 14.6.

Tangent Planes and Normal Lines

If $\mathbf{r} = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is a smooth curve on the level surface $f(x, y, z) = c$ of a differentiable function f, then $f\,(x(t), y(t), z(t))$ is a differentiable function of t. Differentiating both sides (with the help of the Chain Rule and (2)) we obtain

$$\frac{d}{dt}\,f\,(x(t), h(t), k(t)) = \frac{d}{dt}\,c$$

$$\Longrightarrow \nabla f \cdot \frac{d\mathbf{r}}{dt} = 0$$

In other words, at every point along the (smooth) curve, ∇f is orthogonal to the curve's velocity vector. This leads to the following.

Definition. Tangent Plane, Normal Line

The **tangent plane** at $P_0\,(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f\,(P_0)$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f\,(P_0)$.

It follows from chapter 12 that **Tangent Plane** to $f(x, y, z) = c$ at $P_0\,(x_0, y_0, z_0)$ is given by

(3) $$f_x\,(P_0)\,(x - x_0) + f_y\,(P_0)\,(y - y_0) + f_z\,(P_0)\,(z - z_0) = 0$$

and the **Normal Line** to $f(x, y, z) = c$ at $P_0\,(x_0, y_0, z_0)$ is given by the parametric equations

(4) $$x = x_0 + f_x\,(P_0)\,t, \quad y = y_0 + f_y\,(P_0)\,t, \quad z = z_0 + f_z\,(P_0)\,t$$

Example 1. Given the equation of the surface

$$x^2 + 2xy - y^2 + z^2 = 7$$

and the point $Q_0 = Q_0(1, -1, 3)$.

a. Find the equation of the tangent plane at Q_0 on the given surface.

Let $g(x, y, z) = x^2 + 2xy - y^2 + z^2 - 7$. Then

$$\nabla g = (2x + 2y)\ \mathbf{i} + (2x - 2y)\ \mathbf{j} + 2z\ \mathbf{k} \implies$$

$$\nabla g\,(Q_0) = 4\ \mathbf{j} + 6\ \mathbf{k}$$

It follows that the equation of the tangent plane is given by

$$4(y + 1) + 6(z - 3) = 0$$

b. Find the normal line at Q_0 on the surface.

This is easy.

$$x = 1$$

$$y = -1 + 4t$$

$$z = 3 + 6t$$

Standard Linear Approximation

In section 14.3 we discussed the following (two-dimensional) definition of the "total" derivative.

Definition. Let $f : D \subset \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be an interior point of D. Then f is **differentiable** at (x_0, y_0) if there are two numbers $f_1(x_0, y_0)$ and $f_2(x_0, y_0)$ such that

(5)
$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_1(x_0,y_0)(x-x_0) - f_2(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

Later we observed that $f_1 = f_x$ and $f_2 = f_y$. Now let

$$L(x,y) = f(x_0,y_0) + f_1(x_0,y_0)(x-x_0) + f_2(x_0,y_0)(y-y_0),$$

then (5) says that f is differentiable at (x_0, y_0) if there is a linear function $L(x, y)$ such that

(6)
$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - L(x,y)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0$$

We know from experience that if the limit in (6) exists, then $L(x, y)$ is "close" to $f(x, y)$ whenever (x, y) is close to (x_0, y_0). Just as we did in calculus I, we can now define the linearization of a differentiable function f.

Definition. Linearization

Suppose the $f(x, y)$ is a differentiable function. Then the **linearization** of $f(x, y)$ at (x_0, y_0) is the function

(7) $$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

The approximation

(8) $$f(x, y) \approx L(x, y)$$

is called **the standard linear approximation** of f at (x_0, y_0). It is a good approximation of f for all (x, y) "near" (x_0, y_0).

Definition. The Error in the Standard Linear Approximation

The **error** in the approximation defined in (8) is denoted by $E(x, y)$. That is,

$$E(x, y) = f(x, y) - L(x, y)$$

It turns out that we can find an upper bound for this error.

Suppose that f and its first and second partials are continuous in a region containing a rectangle R centered at (x_0, y_0). Suppose also that M is an upper bound on R for $|f_{xx}|, |f_{yy}|,$ and $|f_{xy}|$. Then

(9) $$|E(x, y)| \leq \frac{M}{2}(|x - x_0| + |y - y_0|)^2$$

Example 2. Let $f(x, y) = e^x \sin y$.

a. Find the linearization $L(x, y)$ of f at $P_0 = P_0(\ln 2, \pi/2)$.

$$f_x = e^x \sin y, \implies \quad f_x(\ln 2, \pi/2) = 2$$
$$f_y = e^x \cos y, \implies \quad f_y(\ln 2, \pi/2) = 0$$

so that

$$\begin{aligned} L(x, y) &= f(\ln 2, \pi/2) + f_x(\ln 2, \pi/2)(x - \ln 2) + f_y(\ln 2, \pi/2)(y - \pi/2) \\ &= 2 + 2(x - \ln 2) \end{aligned}$$

b. Find an upper bound for the magnitude $|E|$ of the error in the approximation $f(x, y) \approx L(x, y)$ over the rectangle $R\colon |x - \ln 2| \leq 0.1,\ |y - \pi/2| \leq 0.2$.

The error is bounded by the formula

$$|E| \leq \frac{M}{2}(|x - \ln 2| + |y - \pi/2|)^2$$

where M is an upper bound of *all* of the second order partials of f over the rectangle R. Now,

$$f_{xx} = e^x \sin y \implies |f_{xx}| = |e^x \sin y| \leq e^x \leq e^{\ln 2 + 0.1}, \quad (x, y) \in R$$

and since

$$f_{yy} = -e^x \sin y$$
$$f_{xy} = f_{yx} = e^x \cos y$$

we conclude that $M = e^{\ln 2 + 0.1}$. Thus

$$
\begin{aligned}
|E| &\le \frac{e^{\ln 2+0.1}}{2}(0.1+0.2)^2 \\
&= \frac{e^{\ln 2+0.1}}{2}(0.09) \\
&\le \frac{2.4}{2}(0.09) = 0.108
\end{aligned}
$$

c. Use the linearization of $f(x,y)$ from part (a) to estimate $f(0.75, 1.5)$.
We have

$$
\begin{aligned}
f(0.75, 1.5) &\approx L(0.75, 1.5) \\
&= 2 + 2(0.75 - \ln 2) \\
&\approx 2 + 2(0.75 - 0.693) \\
&\approx 2.114
\end{aligned}
$$

According to MMA $f(0.75, 1.5) \approx 2.1117 \pm 0.0001$.

Differentials and the Derivative

Let $y = f(x)$ be a function and let Δx represent the change in x. Then the corresponding change in y is given by

$$\Delta y = f(x + \Delta x) - f(x)$$

In a first semester calculus class we saw that if f was differentiable at a then

$$\Delta y = \Delta f = f'(a)\Delta x + \varepsilon \Delta x, \qquad \text{where } \ \varepsilon \longrightarrow 0 \ \text{ as } \ \Delta x \longrightarrow 0$$

Now suppose that $z = f(x, y)$ and suppose that x changes from x_0 to $x_0 + \Delta x$ and y changes from y_0 to $y_0 + \Delta y$. Then the corresponding change in z is

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

This leads to the following definition.

Definition. If $z = f(x, y)$, then f is differentiable at (x_0, y_0) if Δz can be expressed in the form

(10) $$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

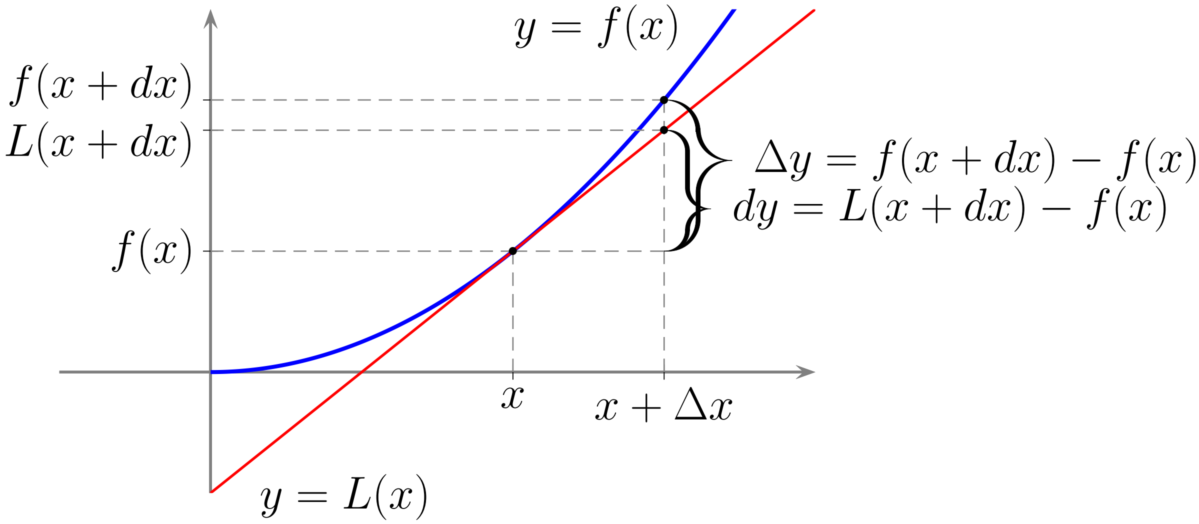
where $\varepsilon_1 \longrightarrow 0$ and $\varepsilon_2 \longrightarrow 0$ as $(\Delta x, \Delta y) \longrightarrow (0, 0)$.

In other words, f is differentiable at (x_0, y_0) if the standard linearization is a good approximation of f “near” (x_0, y_0).

We recall the differential from first semester calculus. Let $y = f(x)$ be differentiable. Then the differential of y (or of f) is given by

(11) $$dy = f'(x)dx$$

The sketch below is helpful.



As we saw with the standard linear approximation,

$$\Delta y \approx dy$$

provided $dx = \Delta x$ is small.

Now suppose that $z = f(x, y)$ and let dx and dy be independent variables. We define the **differential** dz by

(12) $$dz = f_x(x, y)\, dx + f_y(x, y)\, dy = \frac{\partial z}{\partial x}\, dx + \frac{\partial z}{\partial y}\, dy$$

Observe that (8) can now be rewritten as

$$f(x, y) \approx f(x_0, y_0) + dz$$

Example 3. The height and diameter of a tin can is 6 in and 4 in respectively. Use differentials to estimate the amount tin in the can if the tin is 1/16 of an inch thick?

Solution:

Recall that the volume of a cylinder is given by the formula $V = \pi r^2 h$. Now

$$\begin{aligned} dV &= V_r\, dr + V_h\, dh \\ &= 2\pi r h dr + \pi r^2 dh \end{aligned}$$

Since $dr = 1/16$ and $dh = 1/16 + 1/16$ (top and bottom), it follows that

$$\begin{aligned} dV &= V_r(2,6)\,\frac{1}{16} + V_h(2,6)\,\frac{2}{16} \\ &= \frac{2\pi(2)(6)}{16} + \frac{8\pi}{16} \\ &= 2\pi \ \approx \ 6.28318 \text{ in}^3 \end{aligned}$$

Remark. It's not unreasonable to wonder what's the big deal. After all we can simply carry out the following calculation.

$$\begin{aligned}
\Delta V &= V(r+dr, h+dh) - V(r,h) \\
&= \pi\{(r+dr)^2(h+dh) - r^2h\} \\
&= \pi\{r^2h + r^2dh + 2rhdr + 2rdrdh + dr^2h + dr^2dh - r^2h\} \\
&= \underbrace{\pi(r^2dh + 2rhdr)}_{dV} + \pi(2rdrdh + dr^2h + dr^2dh)
\end{aligned}$$

And once again $dr = 1/16$ and $dh = 2/16$. Thus

$$\begin{aligned}
\Delta V &= 2\pi + \pi\left(\frac{8}{16^2} + \frac{6}{16^2} + \frac{2}{16^3}\right) \\
&\approx 6.28318 + 0.173340 \approx 6.456525
\end{aligned}$$

However, the first calculation is much easier.

Example 4. Let $f(x,y) = 3x^2y - x^3\sqrt{y}$.

a. Find the equation of the plane tangent to the surface $z = f(x,y)$ at $P = P(2,4)$.

Now let $g(x,y,z) = f(x,y) - z$. Then the question can be thought of as finding plane tangent to the level surface $g(x,y,z) = 0$ at the point

$$Q = Q(2,4,f(2,4)) = Q(2,4,32).$$

Now we may proceed as we did in Example 1.

$$\begin{aligned} \nabla g &= g_x\,\mathbf{i} + g_y\,\mathbf{j} + g_z\,\mathbf{k} \\ &= (6xy - 3x^2\sqrt{y})\,\mathbf{i} + \left(3x^2 - \frac{x^3}{2\sqrt{y}}\right)\mathbf{j} - \,\mathbf{k} \end{aligned}$$

and

$$\nabla g(Q) = 24\,\mathbf{i} + 10\,\mathbf{j} - \,\mathbf{k}$$

As we saw on page 2, this vector is normal to the plane tangent to the surface $z = f(x,y)$ at Q. It follows that the equation of the plane is

(13) $$24(x-2) + 10(y-4) - 1(z-32) = 0$$

b. Find the linearization of $f(x,y)$ at P.

We could simply follow the recipe given by (7). However, it is easy to see that this is equivalent to solving equation 13 for z. That is,

$$\begin{aligned} z &= 32 + 24(x-2) + 10(y-4) \\ &= f(P) + f_x(P)(x-2) + f_y(P)(y-4) \\ &= L(x,y) \end{aligned}$$

Example 5. The velocity v of a falling object in the absence of wind resistance is given by $v = \sqrt{2hg}$. If the height h is measured with a relative error of 3% and we use 10 m/s for the acceleration due gravity g (instead of 9.81), use differentials to estimate the maximum relative error when measuring v?

So $dg/g = 0.19/9.81 \approx 0.19/10$ and

$$dv = \sqrt{2}\left(\frac{g\,dh}{2\sqrt{hg}} + \frac{h\,dg}{2\sqrt{hg}}\right)$$

It follows that

$$\begin{aligned}\frac{dv}{v} &= \sqrt{2}\left(\frac{g\,dh}{2\sqrt{hg}}\,\frac{1}{\sqrt{2hg}} + \frac{h\,dg}{2\sqrt{hg}}\,\frac{1}{\sqrt{2hg}}\right)\\ &= \frac{1}{2}\left(\frac{dh}{h} + \frac{dg}{g}\right)\\ &= \frac{0.03 + 0.019}{2}\end{aligned}$$

Example 6. Consider the density formula $\rho = m/v$. If an object has a mass m which is measured with a relative error of 2% and a volume v which is measured with a relative error of 5%, find an upper bound of the relative error of the density ρ.

$$d\rho = \frac{dm}{v} - \frac{m\,dv}{v^2}$$

so that

$$\begin{aligned}\frac{d\rho}{\rho} &= \frac{dm}{v}\frac{v}{m} - \frac{m\,dv}{v^2}\frac{v}{m} \\ &= \frac{dm}{m} - \frac{dv}{v}\end{aligned}$$

It follows that

$$\begin{aligned}\left|\frac{d\rho}{\rho}\right| &= \left|\frac{dm}{m} - \frac{dv}{v}\right| \\ &\le \left|\frac{dm}{m}\right| + \left|\frac{dv}{v}\right| \\ &= 0.02 + 0.05\end{aligned}$$

Why were absolute value signs omitted in Example 5?

Remark. Recall that the triangle inequality states that if a and b are real numbers, then

$$|a+b| \le |a| + |b|$$

Notice that we used the triangle inequality in the penultimate step above.