14.5 The Chain Rule

Functions of Two or Three Variables

Theorem 1. Chain Rule for Functions of Three Independent Variables

If $w = f(x, y, z)$ is differentiable and x, y and z are differentiable functions of t, then w is a differentiable function of t and

(1)
$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

Theorem 2. Chain Rule for Functions of Two Independent Variables and Three Intermediate Variables

If $w = f(x, y, z)$ and $x = g(r, s)$, $y = h(r, s)$ and $z = k(r, s)$ are differentiable functions, then w has partial derivatives with respect to r and s and

(2) $$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r}$$

(3) $$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

Example 1. Application - Implicit Function Theorem

Suppose that $F(x, y, z)$ is differentiable and that the equation

(4) $$F(x, y, z) = C$$

defines z implicitly as a (differentiable) function of x and y (i.e., $z = f(x, y)$ and we may assume that x and y are independent variables). Then, with the help of the chain rule, we may differentiate both sides of (4) with respect to x to obtain

(5) $$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

But $\dfrac{\partial x}{\partial x} = 1$ and $\dfrac{\partial y}{\partial x} = 0$ (since y does not depend on x). Thus (5) reduces to

$$\frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(0) + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

or

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Rearranging we obtain

$$\frac{\partial z}{\partial x} = \frac{-\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial z}} = \frac{-F_x}{F_z}$$

Remark. Also, see the next example about the use of the phrase "independent variables".

Example 2. Exercise 14.5.51 - Modified

If $z = f(x, y)$ is differentiable, with $x = r^2 + s^2$ and $y = 2rs$, find $\partial^2 z/\partial s\,\partial r$. (*Note:* That r and s are independent variables goes without saying.)

By the Chain Rule we have

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r} = 2r\frac{\partial z}{\partial x} + 2s\frac{\partial z}{\partial y} \tag{6}$$

Before proceeding, we should discuss the right-hand side of (6). Specifically, we should ask ourselves which of the expressions are functions of s. By assumption, r is not a function of s, but $\partial z/\partial x$, $\partial z/\partial y$, and s are all (differentiable) functions of s.

To make this all readable, we introduce the operator D_s to mean "take the partial with respect to s". We also factor out a 2. Thus

$$\begin{aligned}
D_s\left(\frac{1}{2}\frac{\partial z}{\partial r}\right) &= D_s\left(r\,\frac{\partial z}{\partial x} + s\,\frac{\partial z}{\partial y}\right) \\
&= rD_s\left(\frac{\partial z}{\partial x}\right) + \underbrace{D_s\left(s\,\frac{\partial z}{\partial y}\right)}_{\text{product}} \\
&= rD_s\left(\frac{\partial z}{\partial x}\right) + \underbrace{D_s\left(s\right)\frac{\partial z}{\partial y} + sD_s\left(\frac{\partial z}{\partial y}\right)}_{\text{after applying the product rule}}
\end{aligned}$$

and since $D_s(s) = 1$, the last expression reduces to

$$= rD_s\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} + sD_s\left(\frac{\partial z}{\partial y}\right) \tag{7}$$

So the question is, how on earth do we calculate

$$D_s\left(\frac{\partial z}{\partial x}\right) \quad \text{and} \quad D_s\left(\frac{\partial z}{\partial y}\right)$$

Perhaps one more notational adjustment might help. Let G and H denote the $\partial z/\partial x$ and $\partial z/\partial y$, resp. Then G is a differentiable function of x and y and

$$\begin{aligned} D_s\left(\frac{\partial z}{\partial x}\right) &= D_s(G) = \frac{\partial G}{\partial s} \\ &= \frac{\partial G}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial G}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{\partial z^2}{\partial x^2}\,2s + \frac{\partial z^2}{\partial y\,\partial x}\,2r \end{aligned}$$

and, similarly

$$\begin{aligned} D_s\left(\frac{\partial z}{\partial y}\right) &= \frac{\partial H}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial H}{\partial y}\frac{\partial y}{\partial s} \\ &= \frac{\partial z^2}{\partial x\,\partial y}\,2s + \frac{\partial z^2}{\partial y^2}\,2r \\ &= \frac{\partial z^2}{\partial y\,\partial x}\,2s + \frac{\partial z^2}{\partial y^2}\,2r \end{aligned}$$

Since by the given assumptions from Example 14.5.7, we may conclude that $\dfrac{\partial z^2}{\partial y\,\partial x} = \dfrac{\partial z^2}{\partial x\,\partial y}$.

Putting all of this together we have

$$\begin{aligned}
\frac{\partial z^2}{\partial s\,\partial r} &= D_s\left(\frac{\partial z}{\partial r}\right) = 2D_s\left(\frac{1}{2}\frac{\partial z}{\partial r}\right) \\
&= 2\left\{rD_s\left(\frac{\partial z}{\partial x}\right) + \frac{\partial z}{\partial y} + sD_s\left(\frac{\partial z}{\partial y}\right)\right\} \\
&= 2\left\{r\left(\frac{\partial z^2}{\partial x^2}\,2s + \frac{\partial z^2}{\partial y\,\partial x}\,2r\right) + \frac{\partial z}{\partial y} + s\left(\frac{\partial z^2}{\partial y\,\partial x}\,2s + \frac{\partial z^2}{\partial y^2}\,2r\right)\right\} \\
&= 2\frac{\partial z}{\partial y} + 4rs\frac{\partial z^2}{\partial x^2} + 4(r^2+s^2)\frac{\partial z^2}{\partial y\,\partial x} + 4rs\frac{\partial z^2}{\partial y^2}
\end{aligned}$$