14.7 Optimization and Extreme Values

The Derivative Tests

Definition. Extreme Values

Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all points (x, y) in an open disk centered at (a, b).

Theorem 1. The First Derivative Test for Local Extreme Values

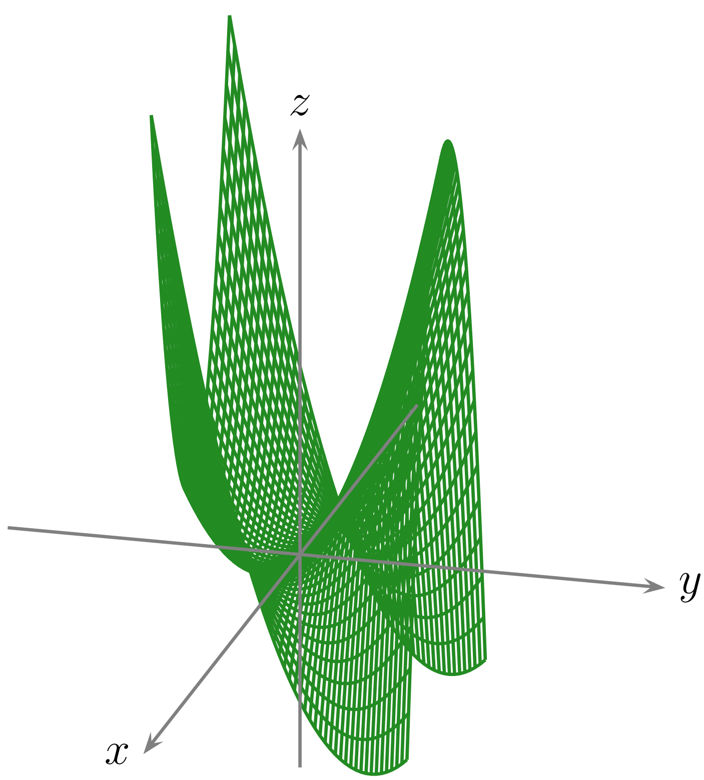
If $f(x, y)$ has a local extreme (min or max) value at an interior point (a, b) of the domain of f and if the partial derivatives exist there, then $f_x(a, b) = f_y(a, b) = 0$.

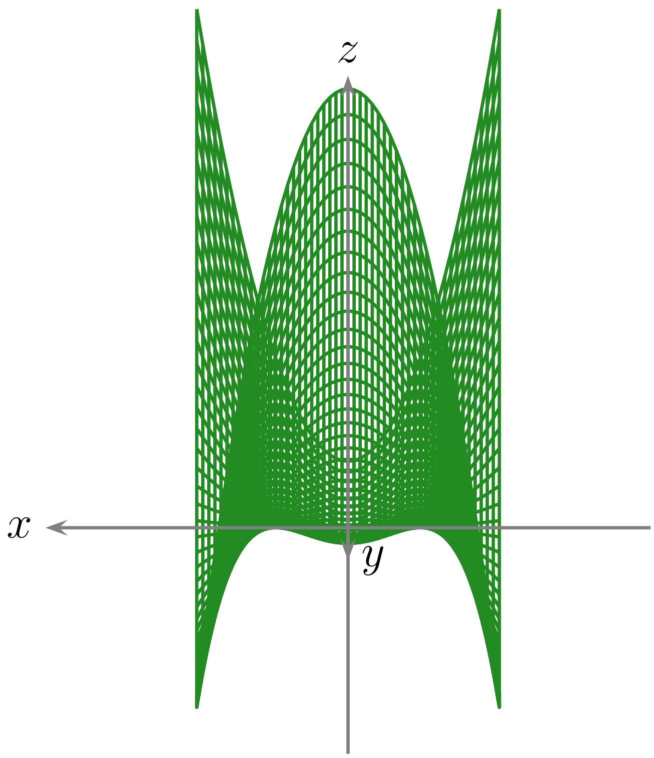
Definition. Critical Point

An interior point of the domain of the function $f(x, y)$ is called a **critical point** of f if either

1. $f_x(a, b) = f_y(a, b) = 0$ or
2. One or both of f_x and f_y do not not exist at (a, b).

Example 1. Sketch of the graph of $z = f(x, y) = y^2 - x^2y + y$. Find the critical points.

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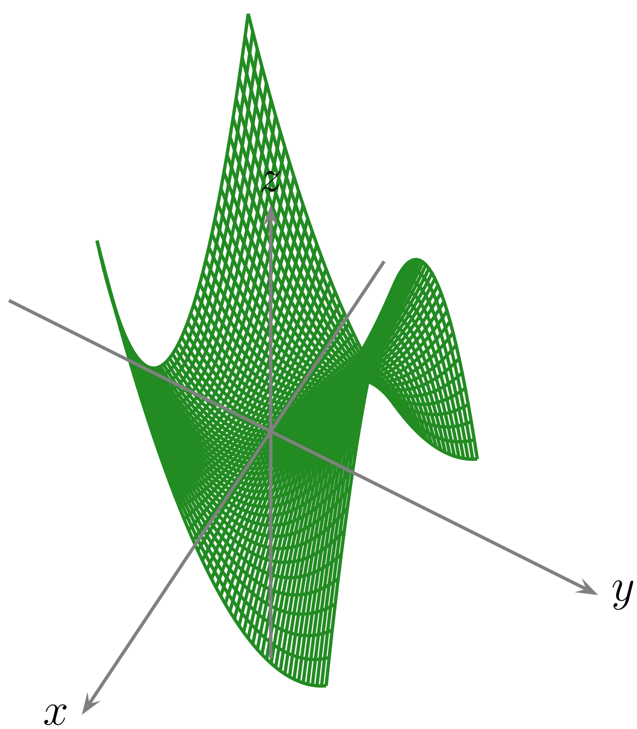
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Definition. Saddle Point

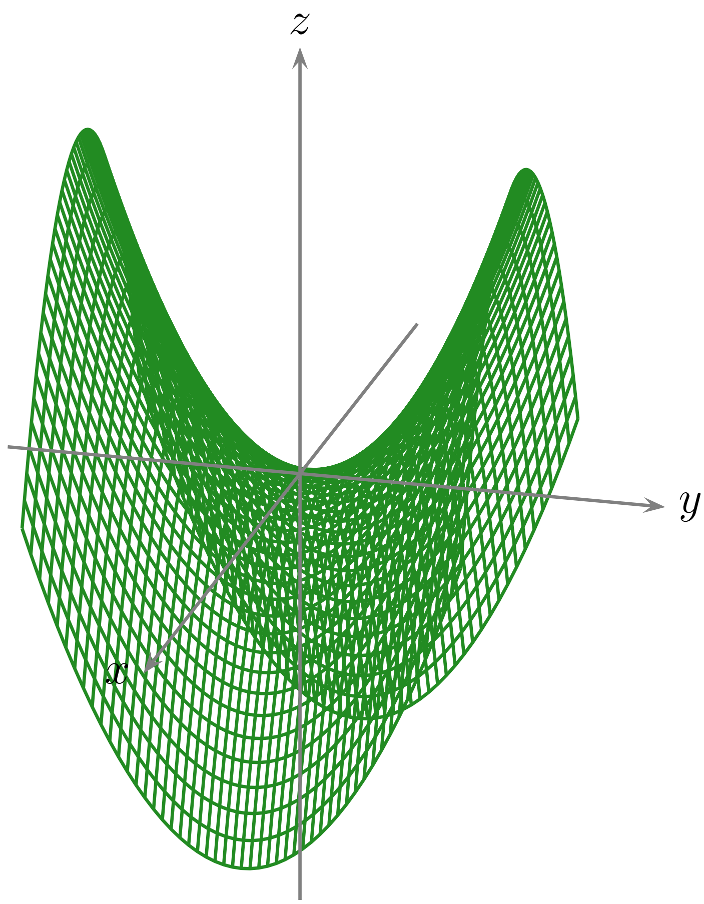
A critical point (a, b) of a differentiable function $f(x, y)$ is called a **saddle point** if in every open disk centered at (a, b) there are domain points (x, y) such that $f(x, y) > f(a, b)$ and other points (x, y) such that $f(x, y) < f(a, b)$.

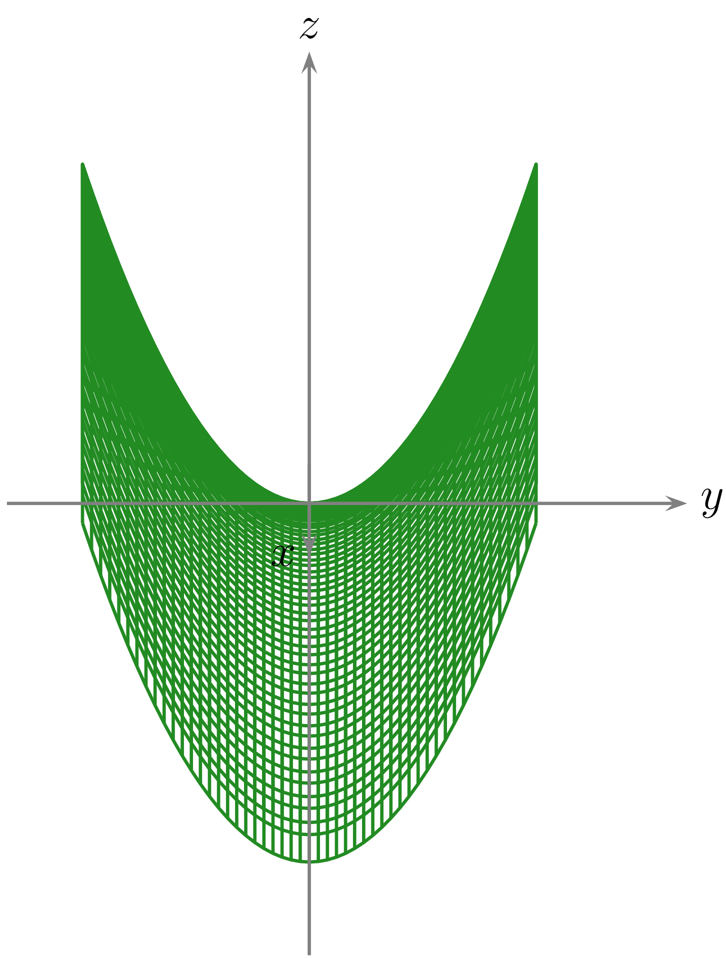
Theorem 2. The Second Derivative Test for Local Extreme Values

Suppose $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

1. f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
2. f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).
3. f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).
4. The test is inconclusive at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b).

Example 2. Sketch of the graph of $z = f(x, y) = \dfrac{y^2 - x^2}{2}$ demonstrating a **saddle point** at the origin.

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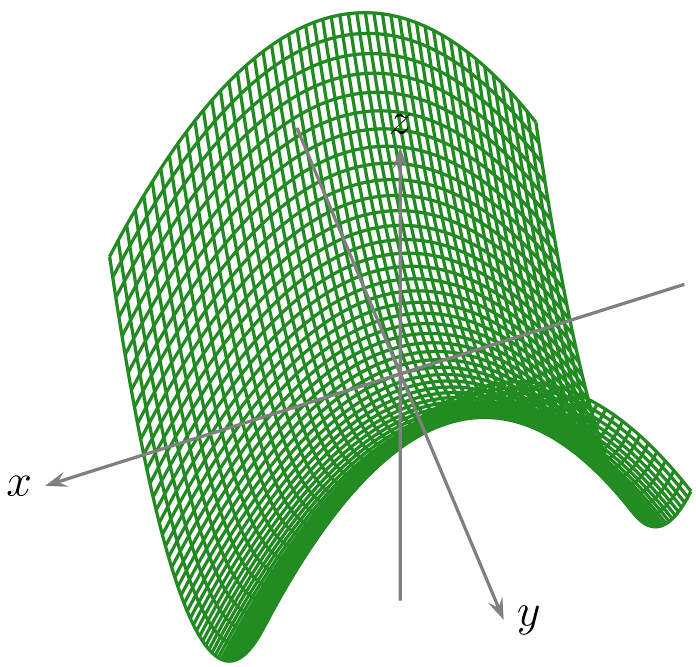
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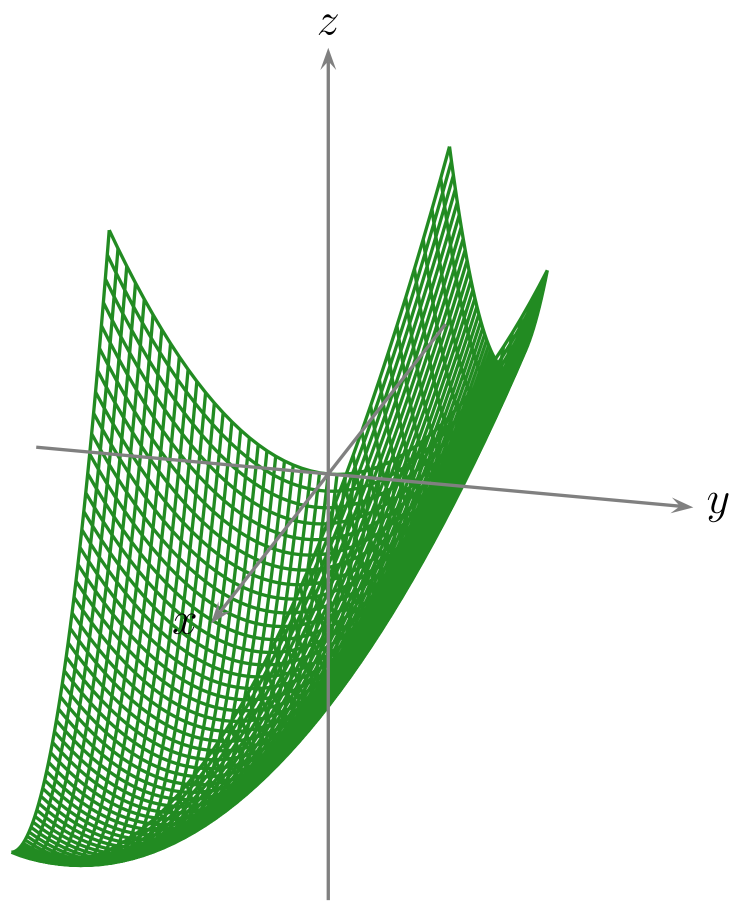
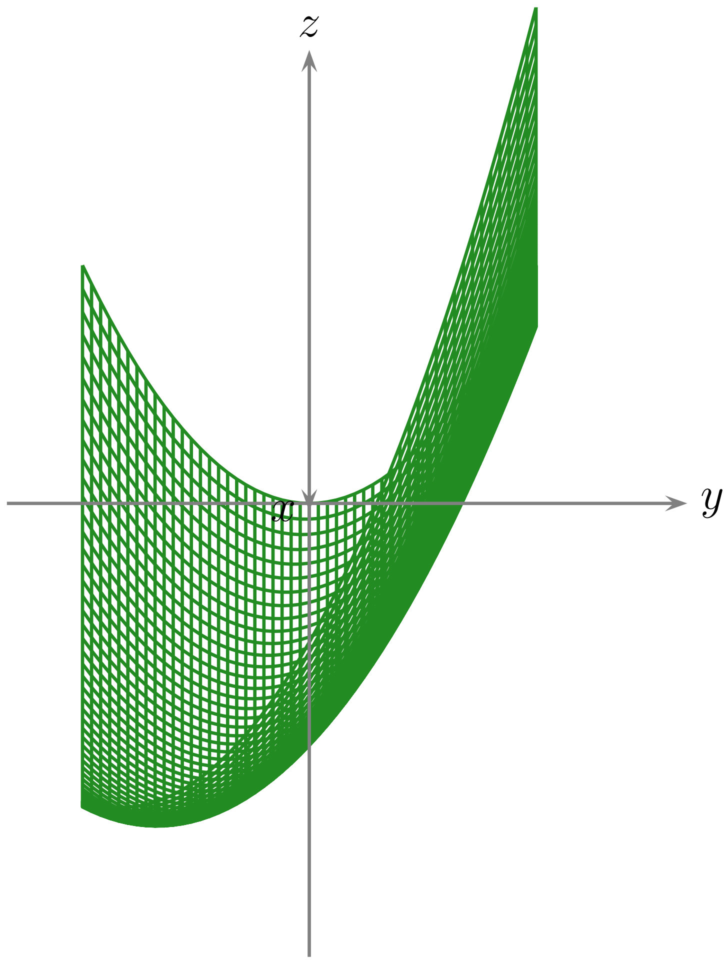
Example 3. Let $z = f(x, y) = x^2 + xy + y^2 - 6x$ and let R be defined by $R : 0 \leq x \leq 5,\ -3 \leq y \leq 3$. Find the maximum and minimum values of f over the region R.

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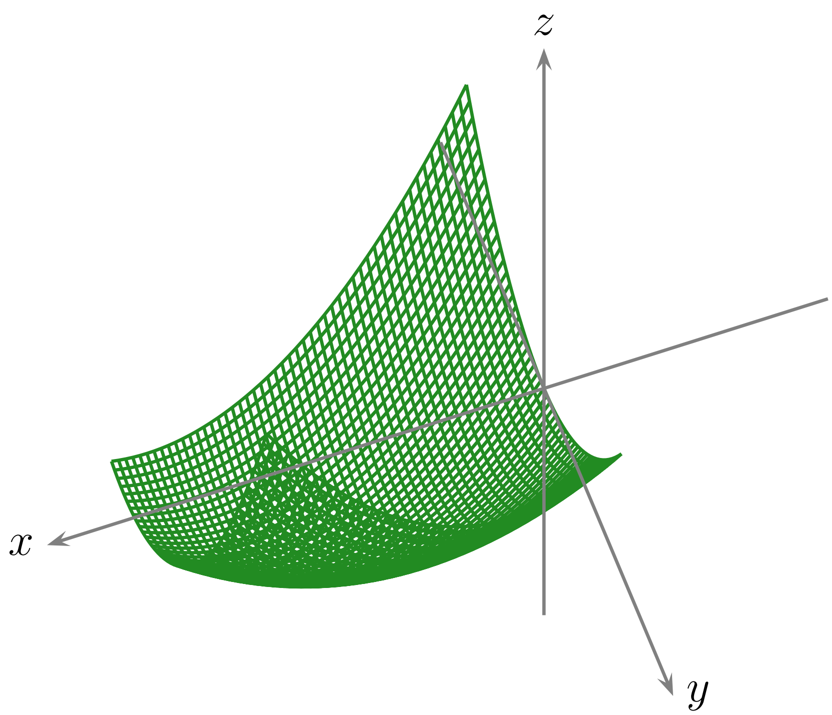
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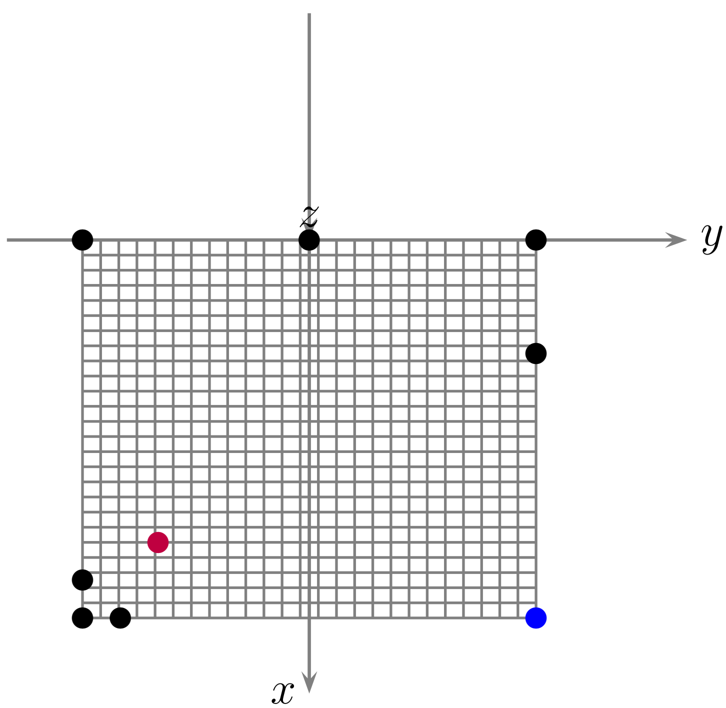
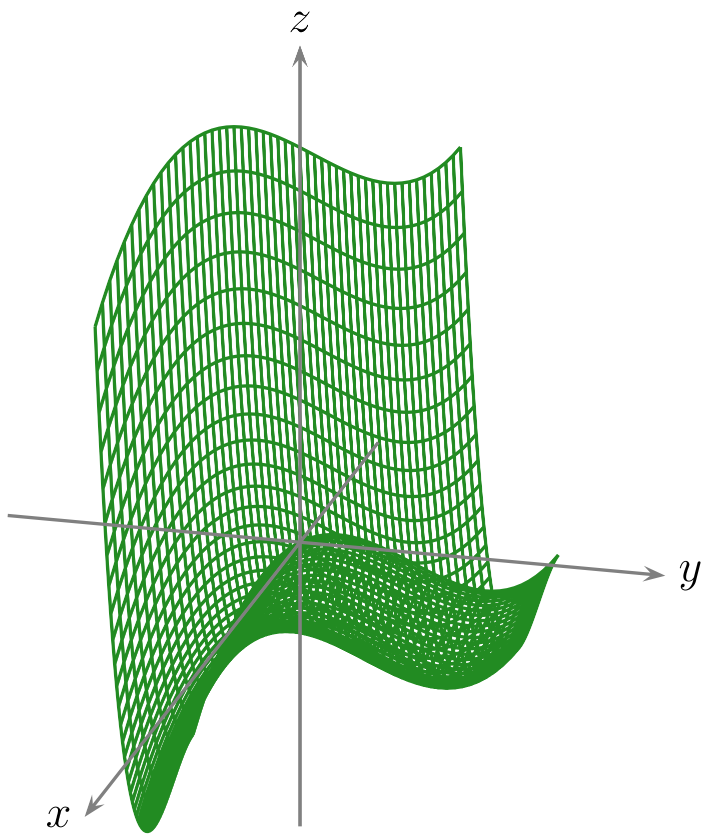
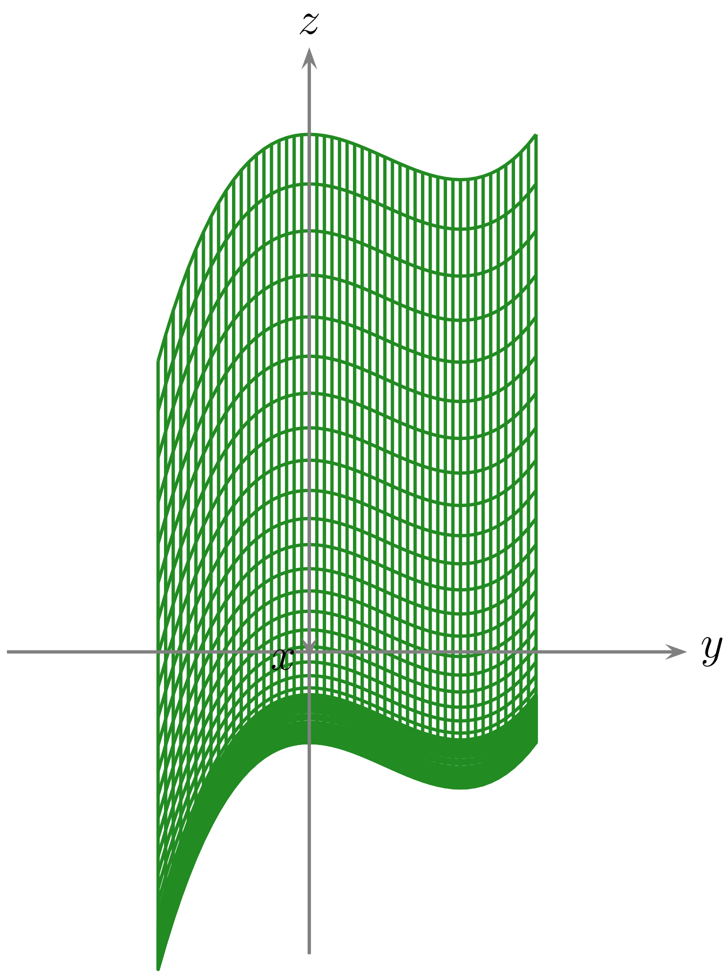
The sketch below indicates the nine domain values that we tested in today's example. The function, $g(x, y)$ attains a global maximum at $(5, 3)$ (shown in blue) and a global minimum at $(4, -2)$ (shown in red).



Example 4. Let $z = f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8$. Find the relative maximum and minimum values of f.

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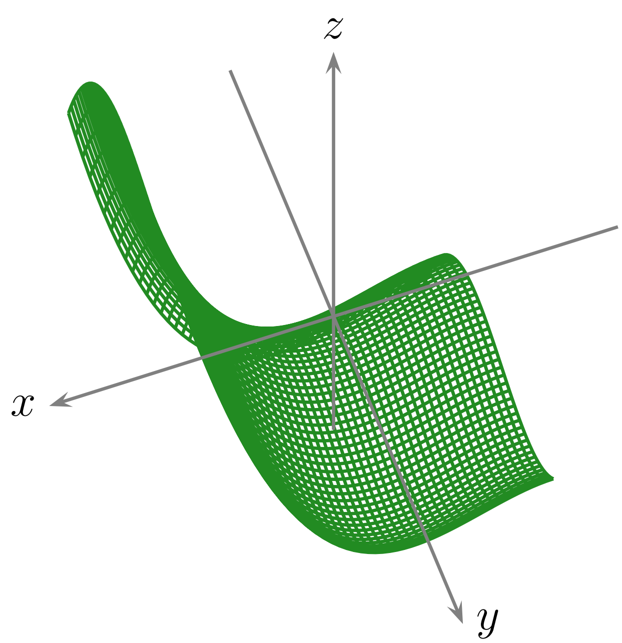
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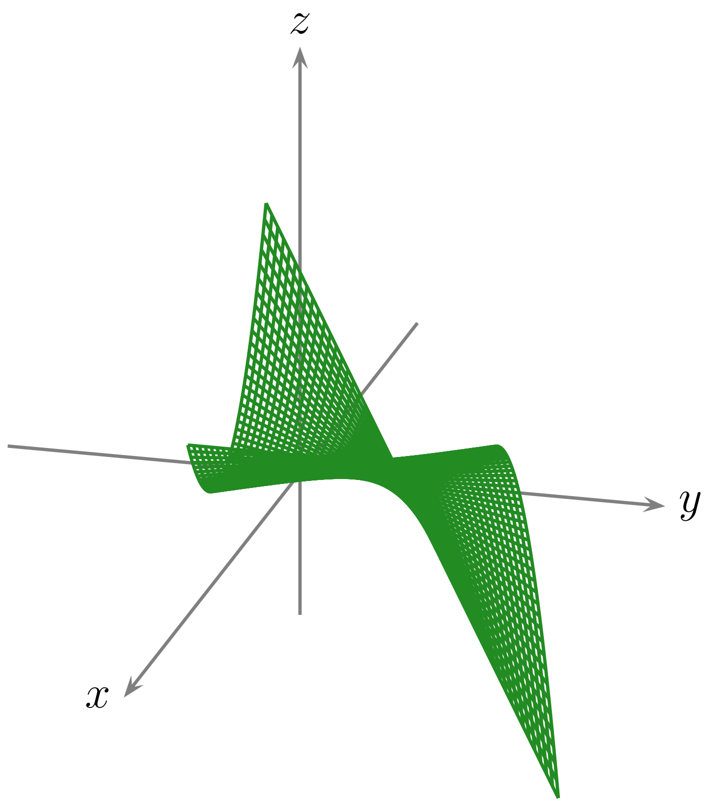
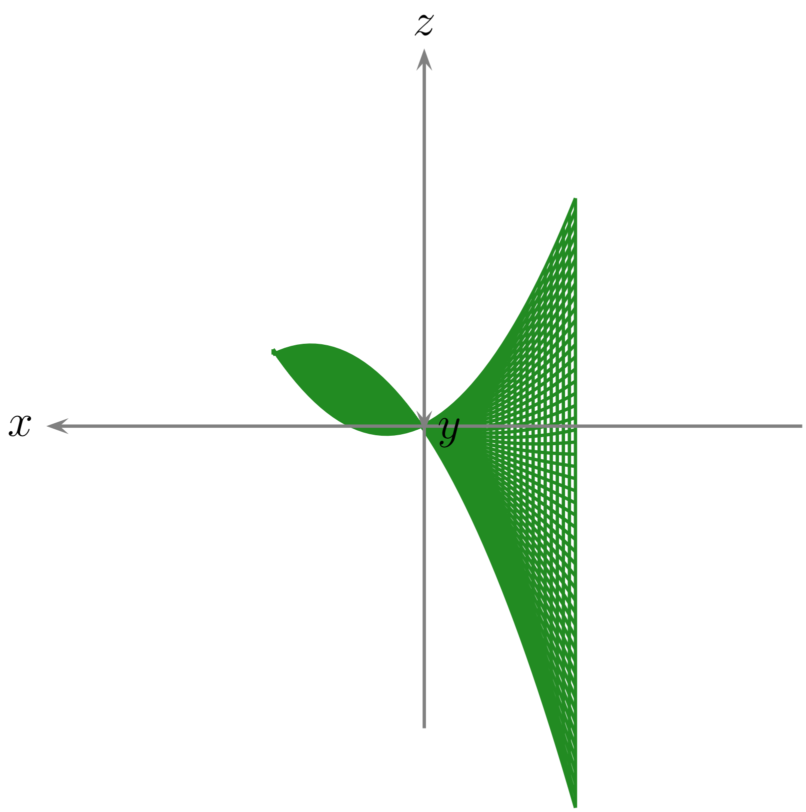
Example 5. Let $z = f(x, y) = x^2 - x^2y + 2xy$ and let R be the region defined by $R : -2 \le x \le 2,\ x^2 - 1 \le y \le 3$. Find the minimum and maximum values of f over the region R. Notice that, unlike the previous two examples, the sketches aren't much help.

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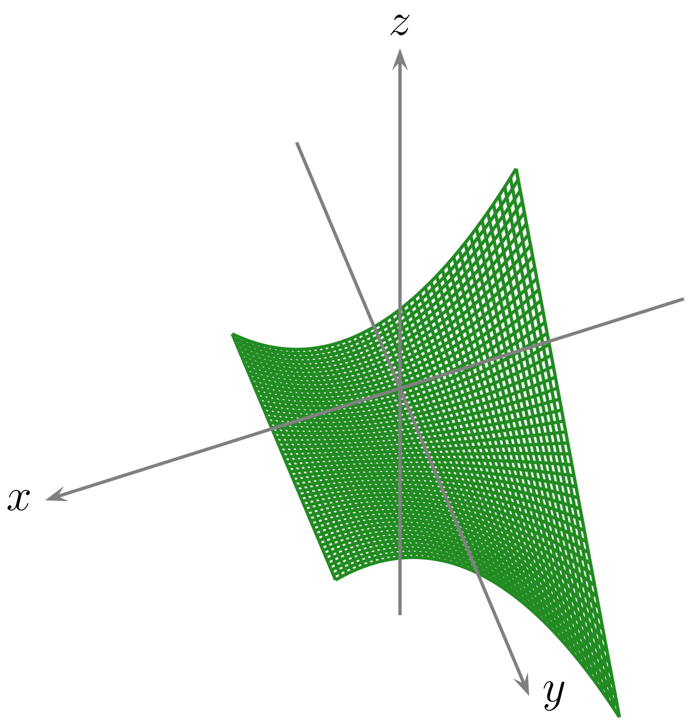
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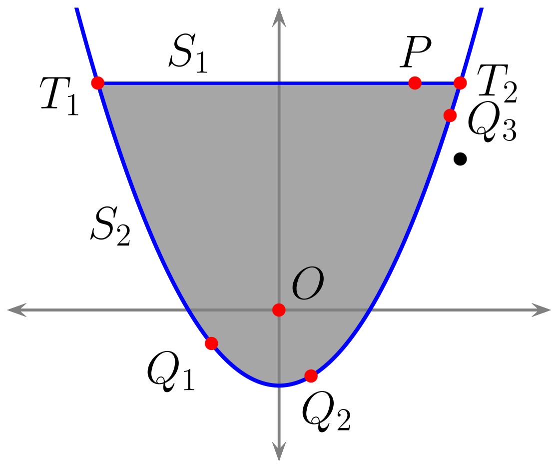
To optimize f, we hope to identify a finite number of domain values to compare. These domain values are shown in the sketch below. Notice that the critical point $(2,2)$ does not lie in R. The procedure to identify these points is outlined below.



Step 1. Find the critical points of f that lie within R.

$$f_x = 2x - 2xy + 2y \quad \text{and} \quad f_y = x(2-x)$$

Setting $f_y = 0$ implies $x = 0, 2$. It follows that the critical points are $O = O(0,0)$ and $(2,2)$.

Step 2. Now let $S_1 = \{(x,3) : -2 \le x \le 2\}$ and let

$$\begin{aligned} h(x) &= f(x,y)\Big|_{S_1} \\ &= f(x,3) = 6x - 2x^2, \quad -2 \le x \le 2 \end{aligned}$$

It is easy to see that 1.5 is a critical point of h. Hence we should add $P = P(1.5, 3)$ to the list of points that we must check.

Step 3. Now let $S_2 = \{(x, x^2 - 1) : -2 \le x \le 2\}$ and let

$$\begin{aligned} g(x) &= f(x,y)\Big|_{S_2} \\ &= f(x, x^2-1) = -2x + 2x^2 + 2x^3 - x^4, \quad -2 \le x \le 2 \end{aligned}$$

With the help of a computer, we identify 3 critical point of g as $-0.744644, 0.355416, 1.889229$. It follows that we must also check

$$Q_1 = Q_1(-0.744644, -0.44550489)$$
$$Q_2 = Q_2(0.355416, -0.8736797)$$
$$Q_3 = Q_3(1.889229, 2.569184)$$

Step 4. Compare the function values at each of the above points and at the points $T_1 = T_1(-2, 3)$ and $T_2 = T_2(2, 3)$.

(x, y)	$f(x, y)$
O	$f(O) = 0$
P	$f(P) = 4.5$
Q_1	$f(Q_1) \approx 1.4650107$
Q_2	$f(Q_2) \approx -0.384355$
Q_3	$f(Q_3) \approx 4.106844$
T_1	$f(T_1) = -20$
T_2	$f(T_2) = 4$

Step 5. Answer the question.

The function attains a minimum at T_1 and a maximum at P.