16.1 Vector Fields

Definition. A **vector field** on a domain (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is a function that assigns a vector to each point in the domain.

For example,

$$\mathbf{F}(x, y, z) = M(x, y, z)\,\mathbf{i} \;+\; N(x, y, z)\,\mathbf{j} \;+\; P(x, y, z)\,\mathbf{k}$$

or

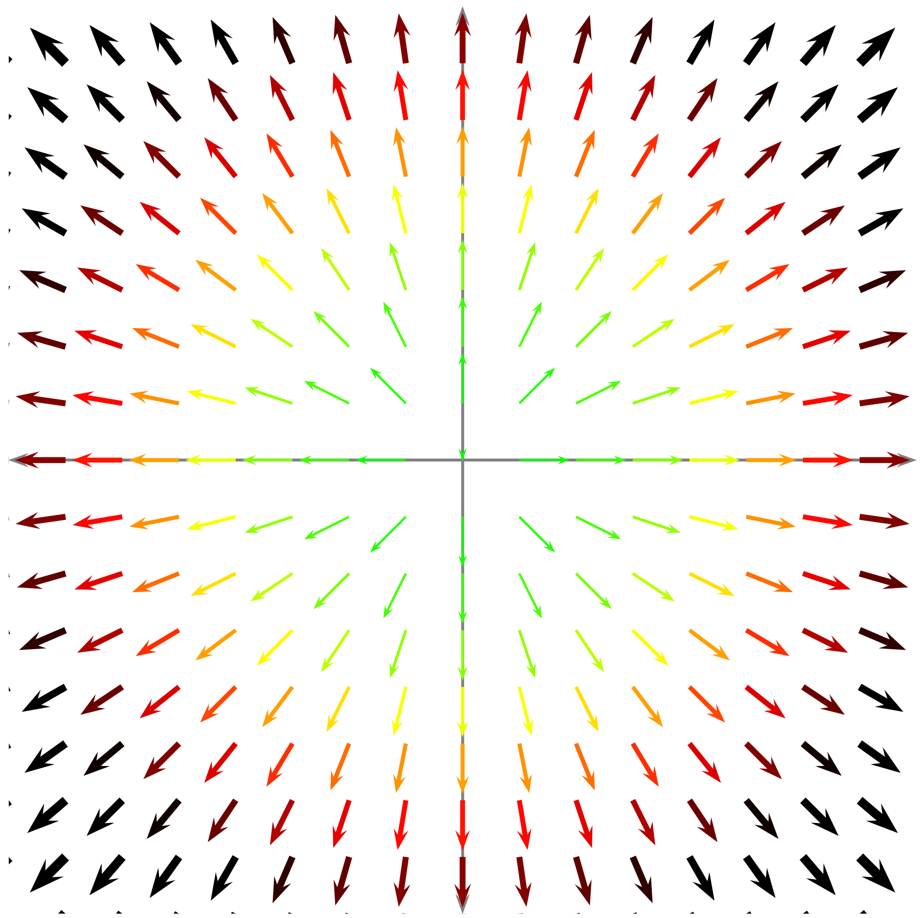
$$\mathbf{F}(x,\, y) = M(x, y)\,\mathbf{i} \;+\; N(x, y)\,\mathbf{j}$$

The field is **continuous** if the component functions M, N, and P are continuous and differentiable if M, N, and P are differentiable.

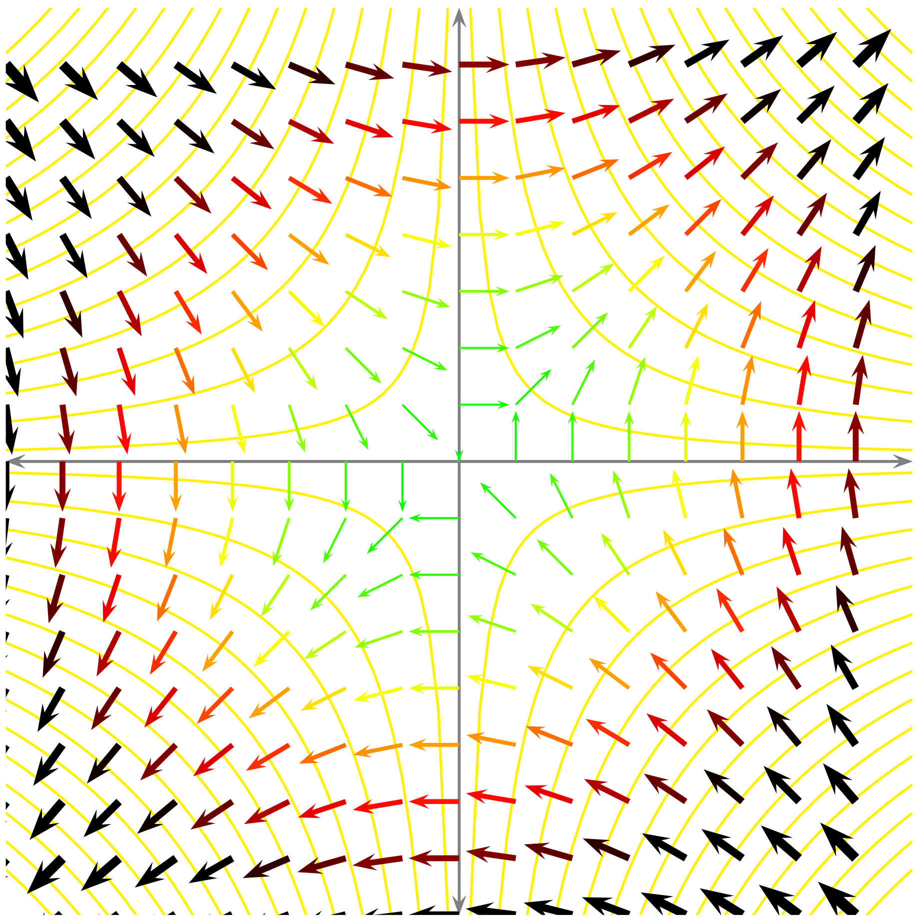
Example 1. Some Vector Fields

(a) The radial field

$$\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j}$$



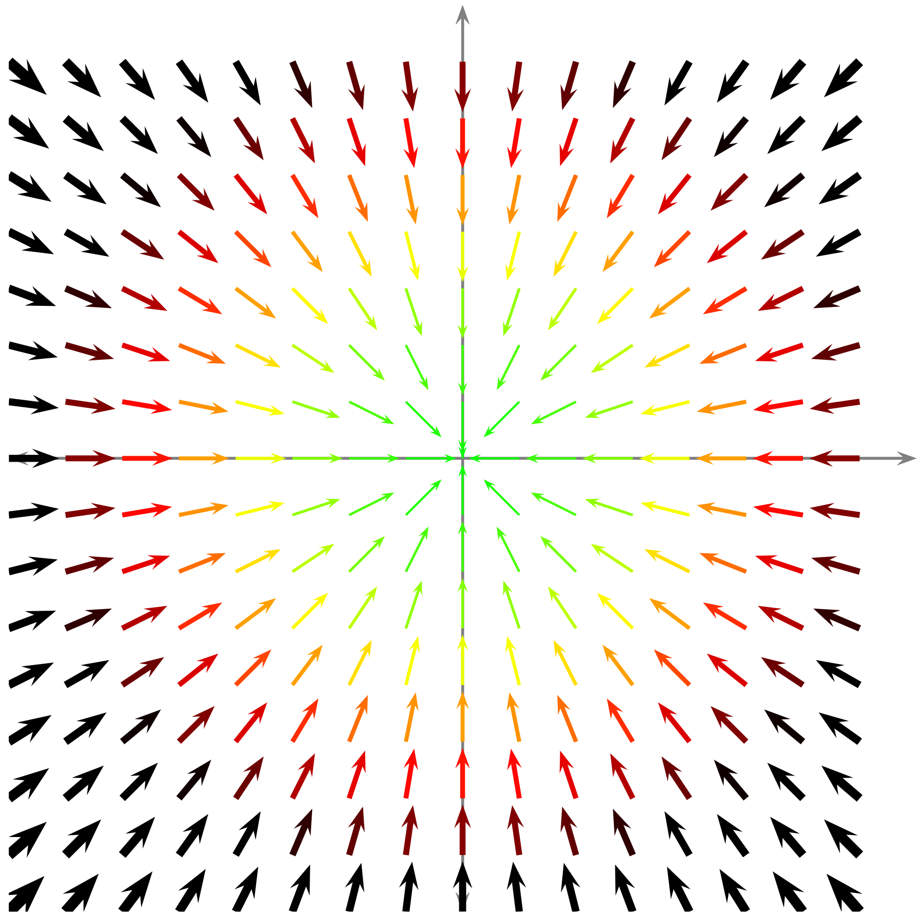
(b) Let $f(x, y) = xy$. Then its gradient vector field $\nabla f = y\,\mathbf{i} + x\,\mathbf{j}$ is shown below. What do the yellow curves represent?



We will have say more about gradient vector fields in subsequent sections.

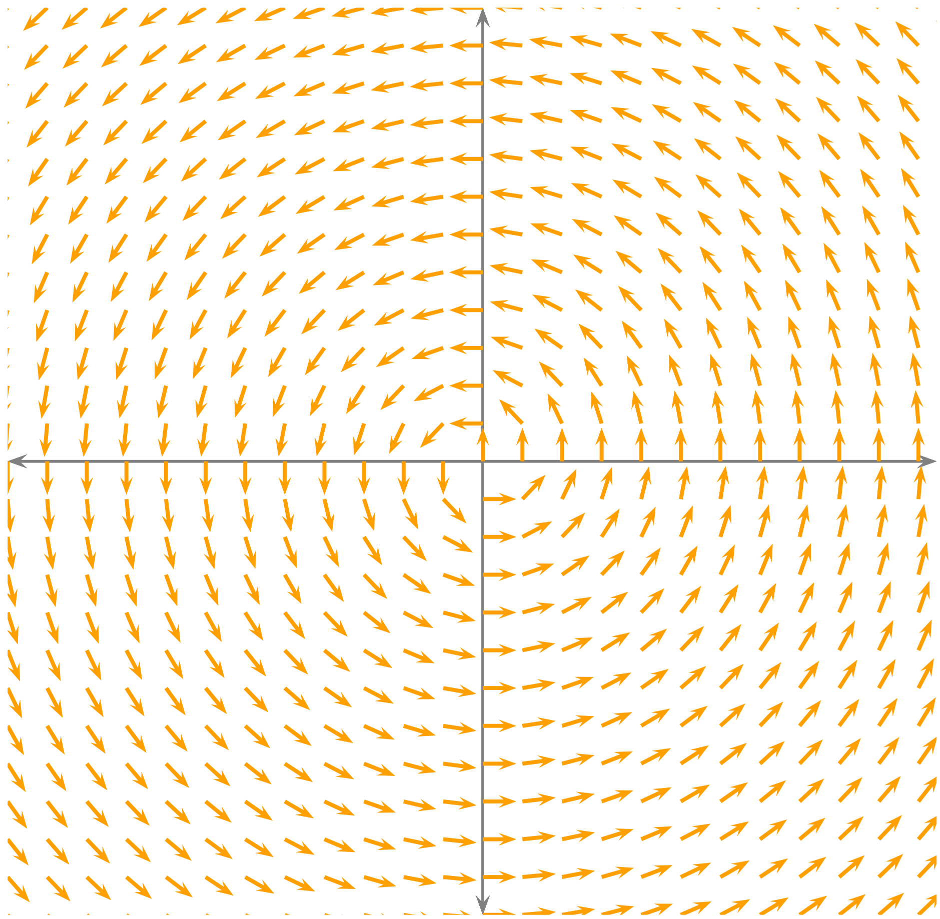
(c) Another radial field

$$\mathbf{F} = -x\,\mathbf{i} - y\,\mathbf{j}$$



(d) The spin field

$$\mathbf{F} = \frac{-y}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{j}$$

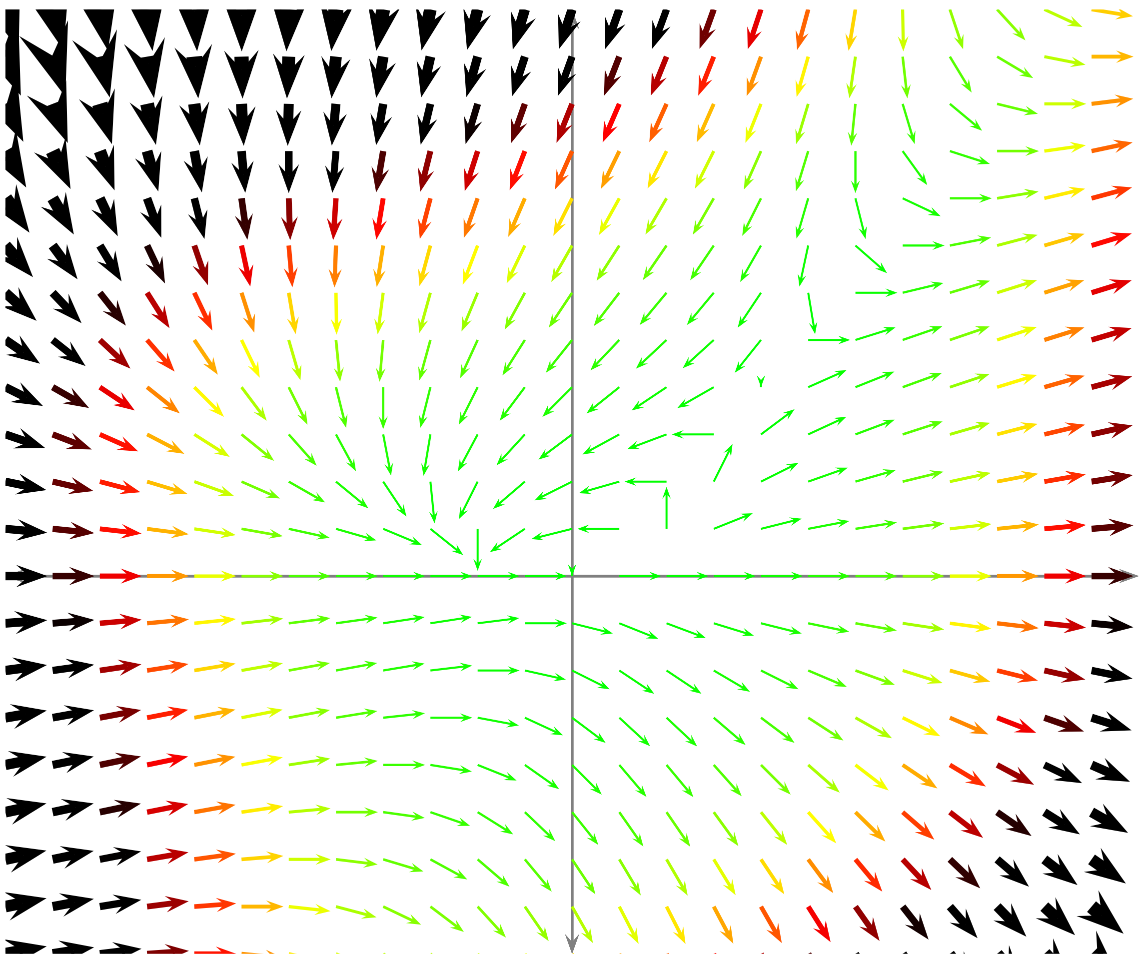


Notice that in this case,

$$M(x,y) = \frac{-y}{\sqrt{x^2+y^2}} \text{ and } N(x,y) = \frac{x}{\sqrt{x^2+y^2}}$$

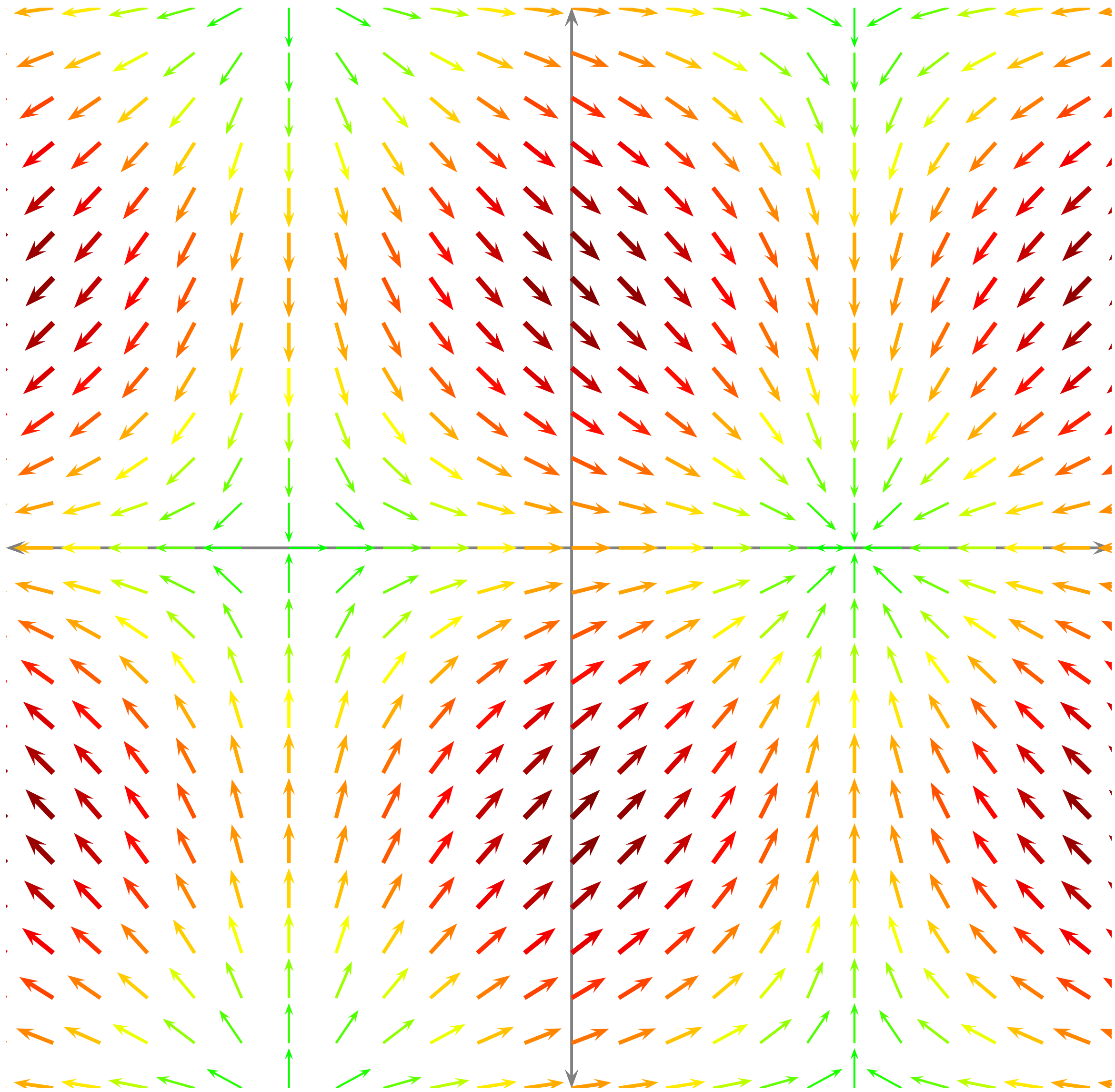
(e) Another vector field

$$\mathbf{F} = (x^2 - y)\,\mathbf{i} + (xy - y^2)\,\mathbf{j}$$



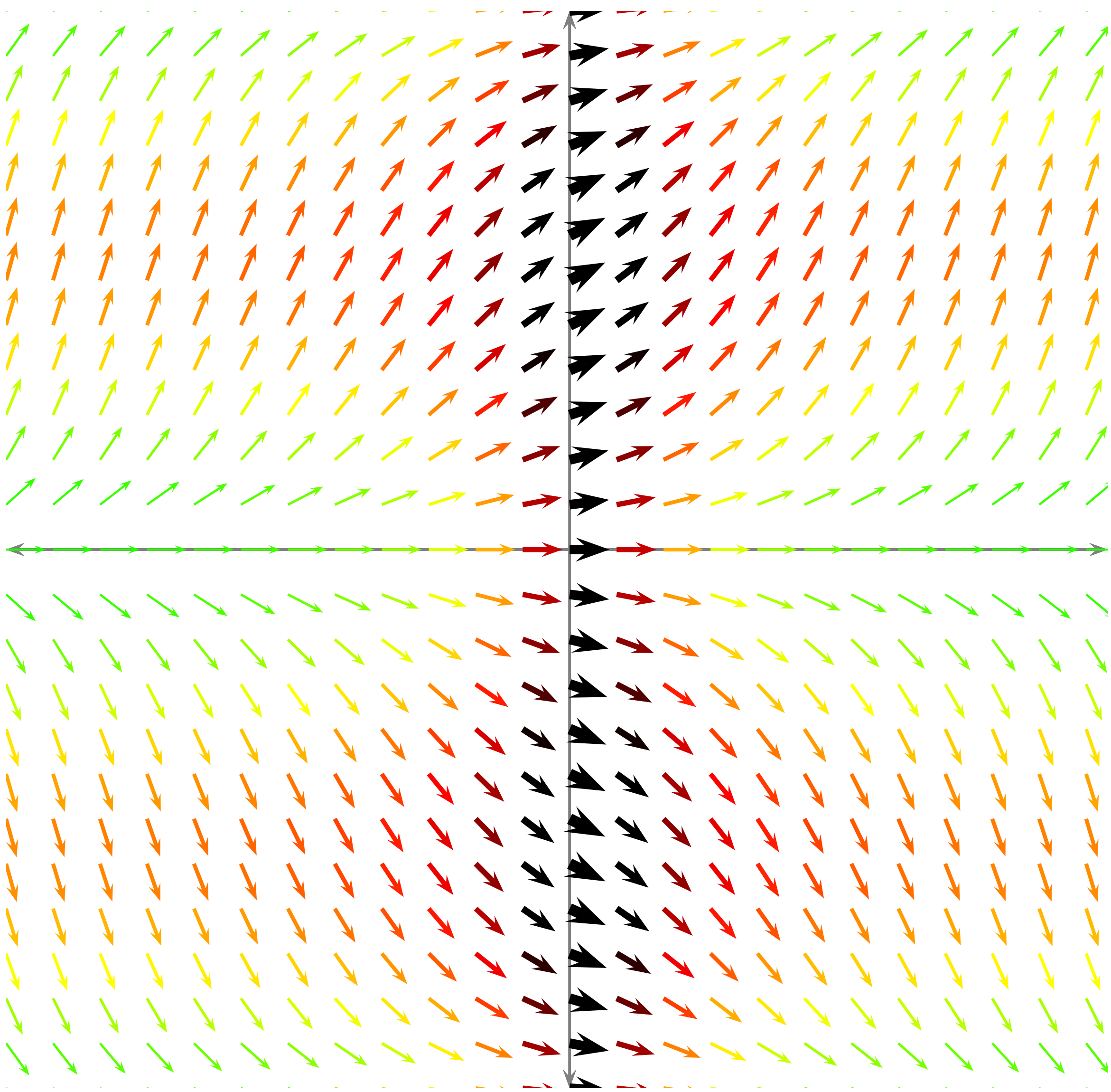
(f) The vector field

$$\mathbf{F} = \cos x\,\mathbf{i} - \sin y\,\mathbf{j}$$



(g) The vector field

$$\mathbf{F} = \frac{2}{2|x|+1}\,\mathbf{i} + \sin y\,\mathbf{j}$$



Definition. The **gradient field** of a differentiable function $f(x, y, z)$ is the field of gradient vectors

$$\nabla f = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j} + \frac{\partial f}{\partial z}\,\mathbf{k}$$

Example 2. A Gradient Field

Let $f(x,\, y) = \sin x + \cos y$. Then the gradient field is

$$\nabla f = \cos x\,\mathbf{i} - \sin y\,\mathbf{j}$$

Definition. A vector field $\mathbf{F}$ is called a **conservative** if there is a function f such that $\nabla f = \mathbf{F}$. In this case, f is called the **potential** function of $\mathbf{F}$. In other words, a vector field $\mathbf{F}$ is conservative if there is a (potential) function f such that $\mathbf{F} = \nabla f$.

Example 3. Let $\mathbf{F} = y^2\,\mathbf{i} + 2xy\,\mathbf{j}$. Then $\mathbf{F}$ is a conservative vector field since $f(x, y) = xy^2$ is a potential function of $\mathbf{F}$. That is

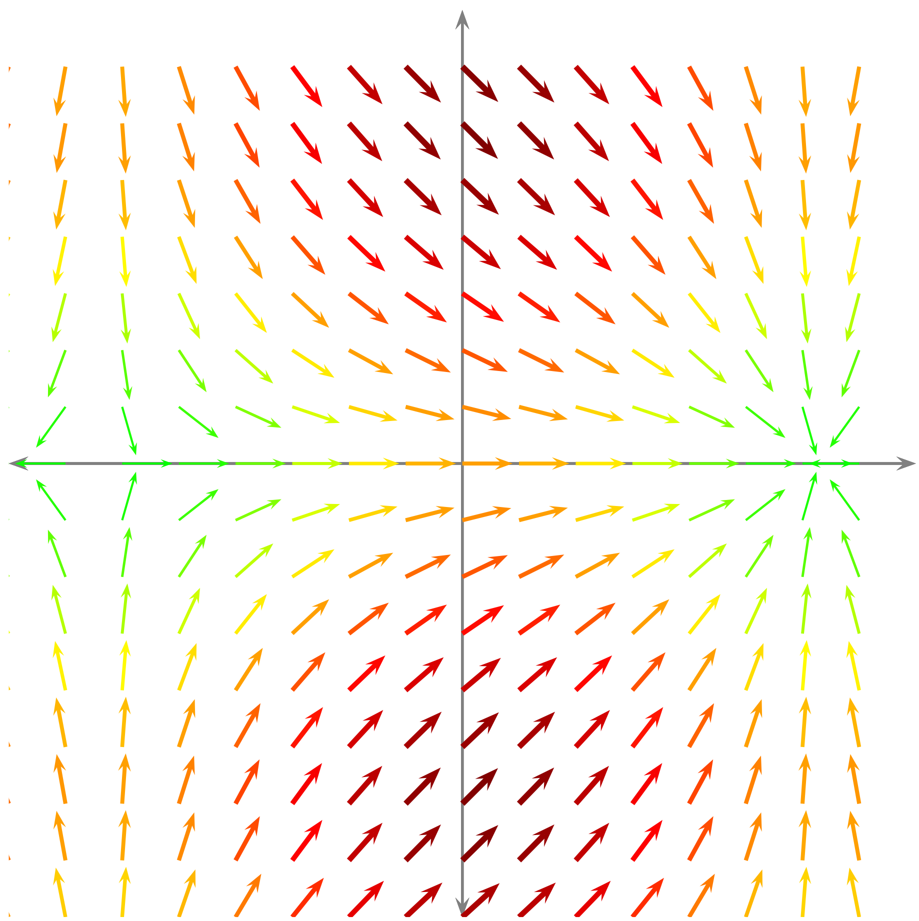
$$\nabla f = \mathbf{F}$$

Remark. It turns out to be important to be able to identify a vector field as the *gradient field* of some function. We will discuss this in more detail in sections 16.3 and 16.5.

Example 4. More Vector Fields

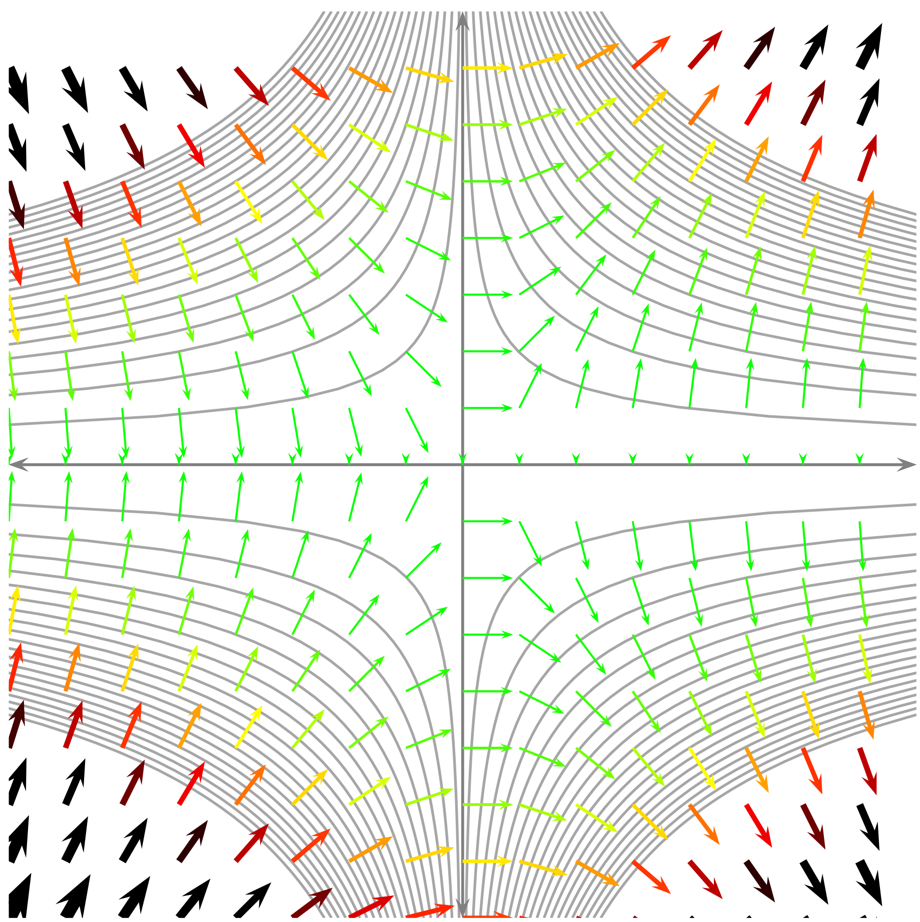
(a) The gradient vector field from Example 2 above. Notice the sink on the positive x-axis.

$$\nabla f = \cos x\,\mathbf{i} - \sin y\,\mathbf{j}$$



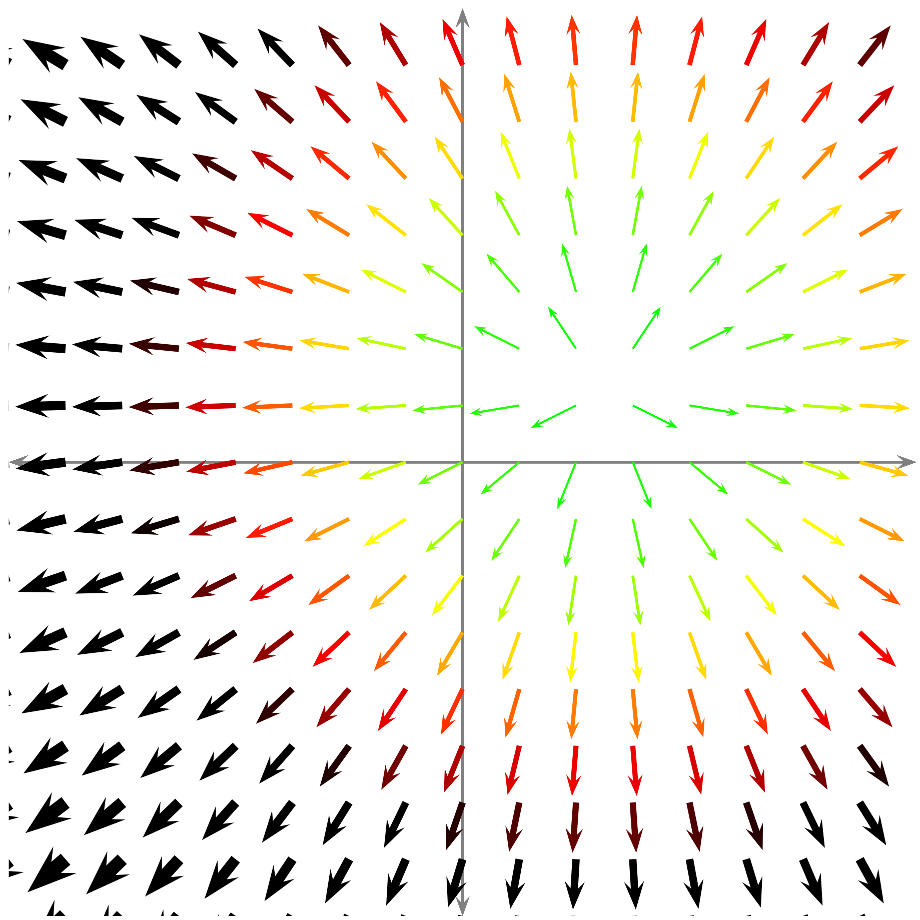
(b) The gradient vector field from Example 3 above. Notice the gradient vectors are everywhere orthogonal to the level curves $f(x,y) = xy^2 =$ const.

$$\nabla f = y^2\,\mathbf{i} + 2xy\,\mathbf{j}$$



(c) A radial field with source at $\left(\frac{1}{2}, \frac{1}{4}\right)$.

$$\mathbf{F} = \left(x - \frac{1}{2}\right)\mathbf{i} + \left(y - \frac{1}{4}\right)\mathbf{j}$$



(d) Looks like a pair of rotational vector fields on both sides of the y-axis.

$$\mathbf{F} = xy\,\mathbf{i} + \frac{1}{1+y^2}\,\mathbf{j}$$