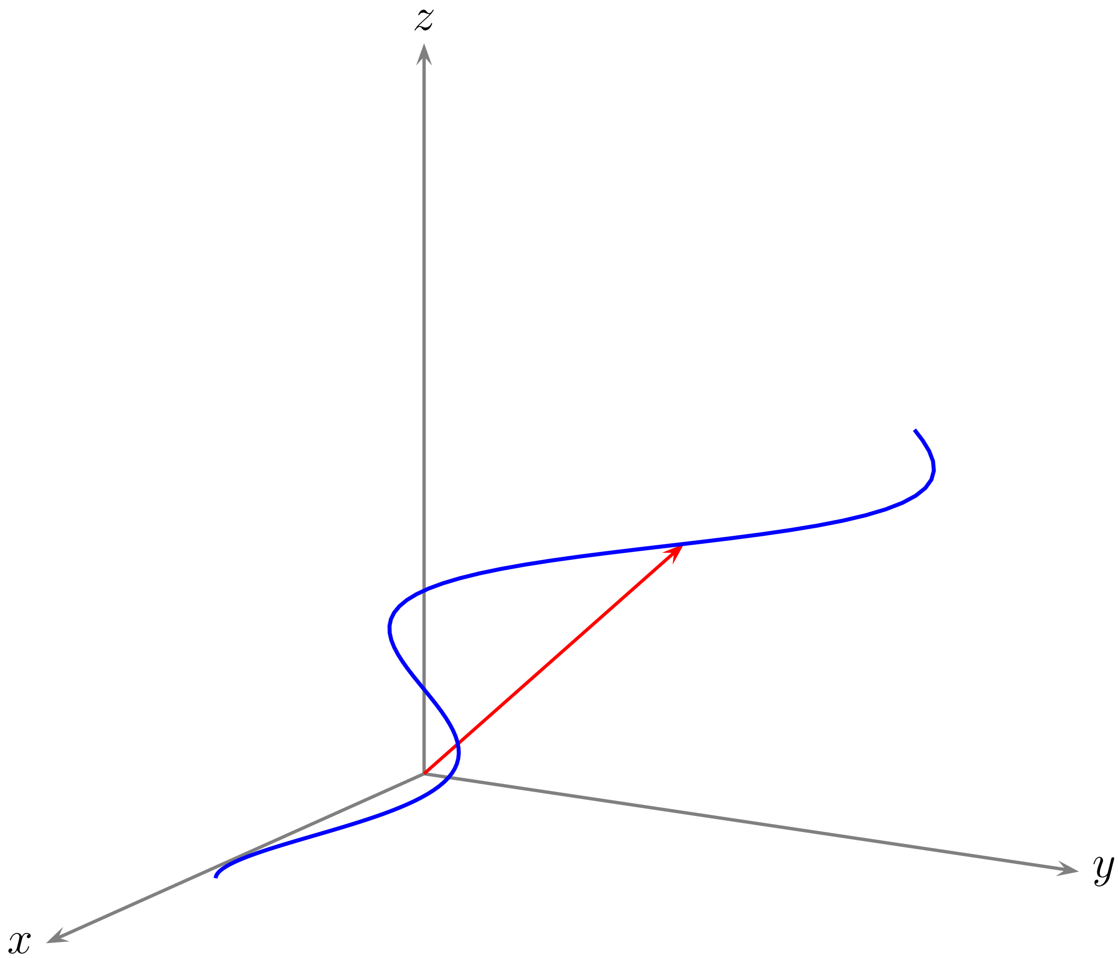
16.2 Line Integrals*



Let $f(x, y, z)$ be defined on a region $D \in \mathbb{R}^3$ containing the **smooth** curve C where C is parameterized by

$$C\colon \mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a \leq t \leq b$$

Recall that C is called a smooth curve if $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$.

* - Some authors also refer to these as “contour integrals”.

Now partition C into a finite number of subarcs (as we have done before) of length Δs_k and form the (Riemann) sum

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k,$$

where (x_k, y_k, z_k) is in the kth subarc. Now if f is continuous and the functions $x,\ y,$ and z have continuous first derivatives, the above sum has a limit as Δs_k approach 0. We call this limit the **(line) integral of** f **over** C **from** a **to** b and denote it by

(1) $$\int_C f(x, y, z)\, ds$$

Now what? If $\mathbf{r}(t)$ is smooth for $a \leq t \leq b$ then

$$s(t) = \int_a^t |\mathbf{r}'(\tau)| \, d\tau$$

Now $\mathbf{r}'(t)$ is continuous, so by the FTC $ds = |\mathbf{r}'(t)| \, dt$ and we have the following:

To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$$

2. We can now evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \, |\mathbf{r}'(t)| \, dt \tag{2}$$

Remark. Stewart initially writes $\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$ instead of $|\mathbf{r}'(t)|\, dt$. Thus (2) is initially written as

$$\int_C f(x, y, z)\, ds = \int_a^b f\left(x(t), y(t), z(t)\right) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt$$

Fortunately, he introduces the equivalent form (2) on page 1092.

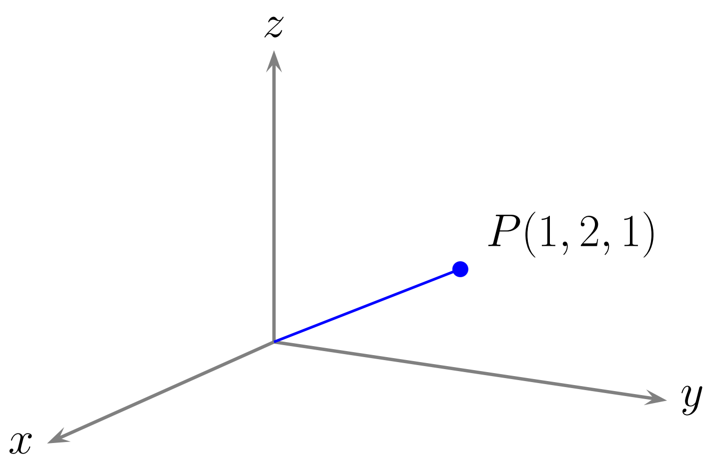
Example 1. Computing a Line Integral

Evaluate the line integral

$$\int_C f(x,y,z)\,ds$$

where $f(x,y,z) = xy + y^3 - z$ and C is

(a) C is the line segment from the origin to $(1,2,1)$.



Let $\mathbf{r}(t) = t\,\mathbf{i} + 2t\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 1$. Then

$$f = 2t^2 + 8t^3 - t$$

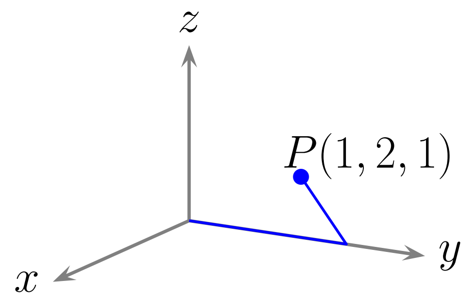
Notice that $\mathbf{r}(t)$ is smooth and

$$|\mathbf{r}'(t)| = \sqrt{(1)^2 + (2)^2 + (1)^2} = \sqrt{6}$$

Thus

$$\begin{aligned}
\int_C f(x, y, z)\,ds &= \sqrt{6}\ \int_0^1 (2t^2 + 8t^3 - t)\,dt \\
&= \sqrt{6}\ \left(\frac{2t^3}{3} + \frac{8t^4}{4} - \frac{t^2}{2}\right)\Bigg|_0^1 \\
&= \sqrt{6}\ \left(\frac{13}{6}\right)
\end{aligned}$$

(b) C is the curve shown in the sketch.



We break up the curve as $C = C_1 \cup C_2$ where

$$C_1: \quad \mathbf{r}_1(t) = 2t\,\mathbf{j}, \quad 0 \le t \le 1; \implies |\mathbf{r}_1'(t)| = 2$$

$$C_2: \quad \mathbf{r}_2(t) = t\,\mathbf{i} + 2\,\mathbf{j} + t\,\mathbf{k}, \quad 0 \le t \le 1; \implies |\mathbf{r}_2'(t)| = \sqrt{2}$$

Thus

$$\int_{C_1 \cup C_2} f(x, y, z)\, ds = 2\int_0^1 (0 + 8t^3 - 0)\, dt + \sqrt{2}\int_0^1 (2t + 8 - t)\ dt$$

$$= 4 + \sqrt{2}\left(\frac{17}{2}\right)$$

Notice that this result differs from the previous one even though we start and end at the same points. More about this later.

Line Integrals of Vector Fields

Suppose the vector field

$$\mathbf{F} = M(x,y,z)\,\mathbf{i} \;+\; N(x,y,z)\,\mathbf{j} \;+\; P(x,y,z)\,\mathbf{k}$$

represents a continuous force field throughout a region in space containing a space curve C that has a smooth parameterization

$$\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}, \quad a \le t \le b$$

We wish to compute the work done by this force in moving a particle along C.

Definition. The **work** done by the force $\mathbf{F}$ over the smooth curve C is given by

(3) $$W = \int_C \mathbf{F}\cdot\mathbf{T}\,ds$$

where $\mathbf{T}$ is the unit tangent vector.

Once we choose a (smooth) parameterization, (3) is usually written as

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_{t=a}^{t=b} \mathbf{F}\cdot\mathbf{T}\,ds$$

Remark. Since $\mathbf{T} = d\mathbf{r}/ds$ we may rewrite (3) as

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{t=a}^{t=b} \mathbf{F}\cdot\mathbf{T}\,ds \\ &= \int_{t=a}^{t=b} \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\,ds \\ &= \int_{t=a}^{t=b} \mathbf{F}\cdot d\mathbf{r}\end{aligned}$$

where $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

In fact, we have several different ways to write the work integral:

$$\begin{aligned}\int_C \mathbf{F}\cdot\mathbf{T}\,ds &= \int_{t=a}^{t=b} \mathbf{F}\cdot\mathbf{T}\,ds \\ &= \int_{t=a}^{t=b} \mathbf{F}\cdot d\mathbf{r} \\ &= \int_a^b \mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\,dt \\ &= \int_a^b \left(M\,\frac{dx}{dt} + N\,\frac{dy}{dt} + P\,\frac{dz}{dt}\right)\,dt \\ &= \int_a^b M\,dx + N\,dy + P\,dz\end{aligned}$$

As we discussed in class, $\mathbf{F}$ need not be a force field. See, for example, the notes on flow and flux starting on page 18. We have the following.

Definition. Line Integral of $\mathbf{F}$ along C

Let $\mathbf{F}$ be a continuous vector field defined on a smooth curve C and suppose that C is parameterized by a vector function $\mathbf{r}(t),\ a \le t \le b$. Then the line integral of $\mathbf{F}$ along C is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_{t=a}^{t=b} \mathbf{F} \cdot d\mathbf{r} \tag{4}$$

Example 2. Evaluating a Work Integral

Let $\mathbf{F} = x^2\ \mathbf{i} - y\ \mathbf{j}$ and let C be the curve from $A(4,\ 2)$ to $B(1,\ -1)$ along the parabola $x = y^2$. See Figure 1 below.

We use the parametrization (see Example 3 below to see how to do this quickly).

(5) $$C\colon\ \mathbf{r}(t) = (2-3t)^2\,\mathbf{i} + (2-3t)\,\mathbf{j}, \quad 0 \le t \le 1$$

a. Evaluate the line integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

First notice that $x = (2-3t)^2$ and $y = (2-3t)$ so that

$$\mathbf{F}(t) = \Big\langle (2-3t)^4, -(2-3t) \Big\rangle$$

Now by (5) we have

$$\frac{d\mathbf{r}}{dt} = \langle -6(2-3t), -3 \rangle$$

It follows that

$$\begin{aligned}\int_{t=0}^{t=1} \mathbf{F}\cdot d\mathbf{r} &= \int_0^1 \Big\langle (2-3t)^4, -(2-3t)\Big\rangle \cdot \langle -6(2-3t), -3\rangle\ dt \\ &= \int_0^1 -6(2-3t)^5 + 3(2-3t)\,dt \\ &\overset{*}{=} \vdots \\ &= -21 + 3/2 = -39/2\end{aligned}$$

* – The actual integration calculations have been suppressed since they are trivial.

b. Find the **work** done by the force field $\mathbf{F}$ on a particle that moves along the curve C from A to B.

By definition this is

$$\begin{aligned}\int_{t=0}^{t=1} \mathbf{F}(t) \cdot \mathbf{T}(t)\, ds &= \int_{t=0}^{t=1} \mathbf{F}(t) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\ |\mathbf{r}'(t)|\, dt \\ &= \int_{t=0}^{t=1} \mathbf{F}(t) \cdot \mathbf{r}'(t)\, dt \\ &= \int_C \mathbf{F} \cdot d\mathbf{r} \\ &= -39/2\end{aligned}$$

by part (a).

c. Evaluate the line integral $\int_C x^2\, dx - y\, dy$. Same as (a) (why?), so

$$\int_C x^2\, dx - y\, dy = -39/2$$

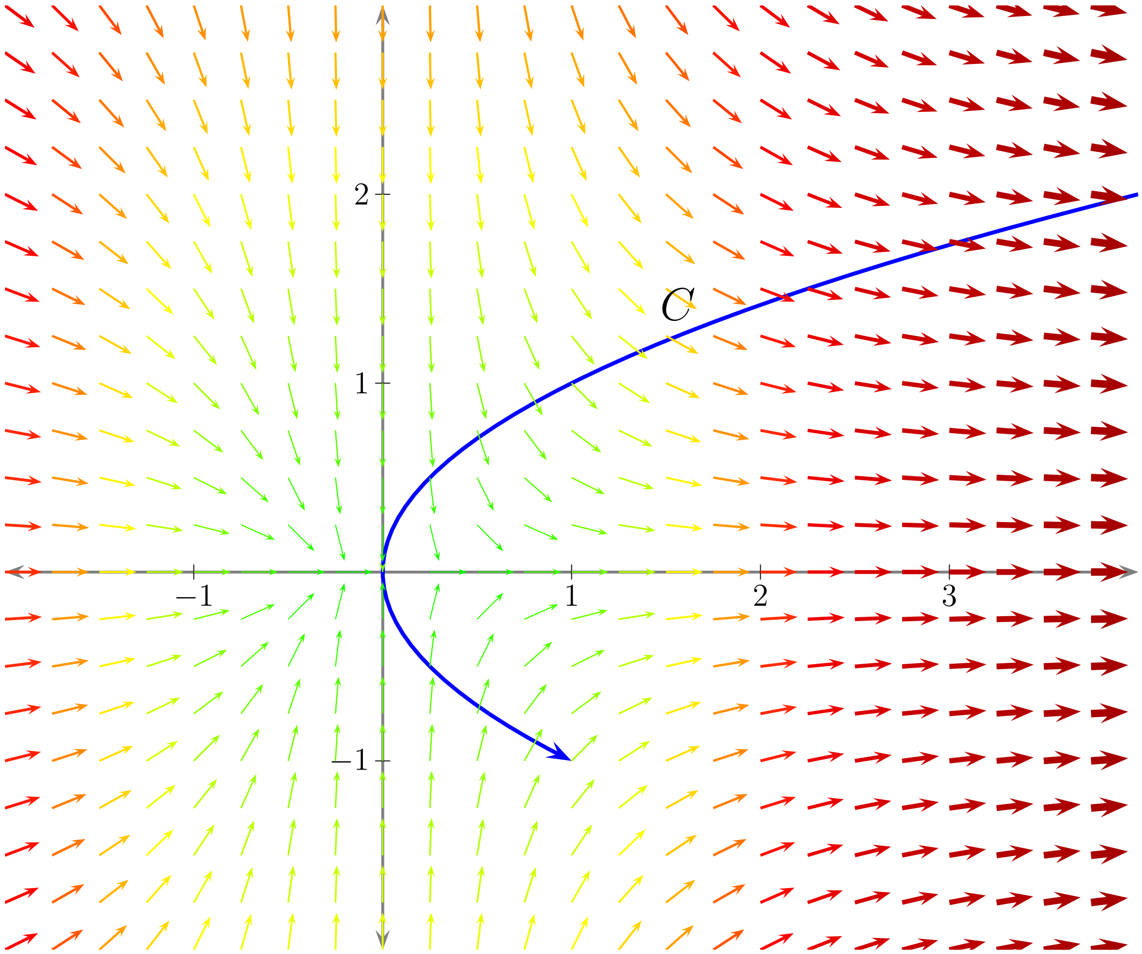


Figure 1: The Force Field: $\mathbf{F} = \langle x^2, -y \rangle$

Example 3. (Re)Parameterizing a Curve.

How to parameterize the curve $y = f(x)$. Suppose that one wishes to parameterize a given curve from $P = (b, f(b))$ to $Q = (a, f(a))$. One choice is to set $x = s$ and $y = f(s)$. (Of course, we make the obvious modifications for the curve $x = g(y)$.) Now

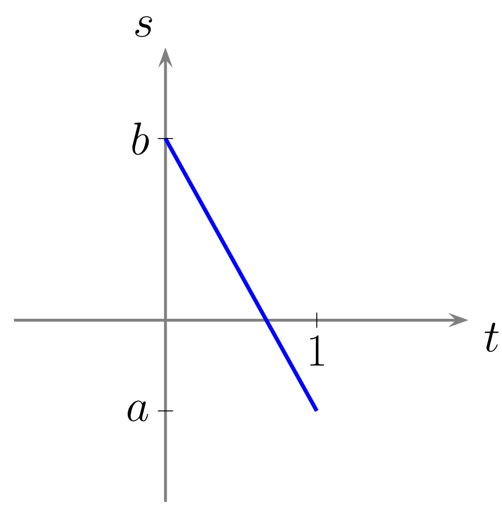
(6) $$\mathbf{r} = s\,\mathbf{i} + f(s)\,\mathbf{j},$$

where s lies between a and b. Now if $b < a$ then we are done. On the other hand, if $a < b$, the parameterization is from Q to P which is not what we want.

Recall, that the expression

$$s = b(1-t) + at, \quad 0 \le t \le 1 \tag{7}$$

yields all real numbers from b to a starting with b. To see this, consider the sketch below.



Notice that the slope of the line in the above sketch is $(a-b)$ so that

$$\begin{aligned} s &= b + (a-b)t \\ &= b(1-t) + at \end{aligned}$$

as we claimed in (7).

So let $s = b(1-t) + a(t)$ in (6). It follows that the desired parameterization is given by

$$x = b(1-t) + at$$

$$y = f(b(1-t) + at), \quad 0 \le t \le 1$$

or

(8) $$\mathbf{r} = (b(1-t) + a(t))\,\mathbf{i} + f(b(1-t) + a(t))\,\mathbf{j}, \quad 0 \le t \le 1$$

As an application, consider the curve $x = y^2$ from the previous example. In this case the obvious parameterization yields

$$y = s,\ x = g(s) = s^2, \quad -1 \le s \le 2$$

but this traces the curve in the wrong direction. Instead, we apply (8) with

$b = 2$ and $a = -1.$

Thus

$$y = 2(1-t) + (-1)t = 2 - 3t$$

$$x = (2-3t)^2, \quad 0 \le t \le 1$$

as we claimed above.

Example 4. A Calculus II Example

Let $a < b$. In second semester calculus we saw that the work done by a variable force $M(x)$ directed along the x-axis from $x = a$ to $x = b$ was given by the definite integral

$$W = \int_a^b M(x)\,dx \tag{9}$$

Show that (9) is a special case of (3).

Consider the vector field $\mathbf{F}(x, y) = M(x)\,\mathbf{i}$. Find the work done by the force $\mathbf{F}$ from $(a, 0)$ to $(b, 0)$ along the x-axis.

Notice that we have the parametrization

$$\mathbf{r}(t) = (a(1-t) + bt)\,\mathbf{i}, \quad 0 \le t \le 1$$

of the line segment. It follows that

$$\mathbf{F}(\mathbf{r}(t)) = M(a(1-t) + bt)\,\mathbf{i}$$

Now $\mathbf{r}'(t) = (b-a)\,\mathbf{i}$ and $|\mathbf{r}'(t)| = b - a$. Thus

$$\begin{aligned} W &= \int_C \mathbf{F} \cdot \mathbf{T}\,ds \\ &= \int_0^1 \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt \\ &= \int_0^1 M(a(1-t) + bt)(b-a)\,dt \end{aligned}$$

Now let $x = a(1-t) + bt$. Then $dx = (b-a)\,dt$ and

$$\begin{aligned} W &= \int_0^1 M(\underbrace{a(1-t) + bt}_{x})\,\underbrace{(b-a)\,dt}_{dx} \\ &= \int_{x(0)}^{x(1)} M(x)\,dx \\ &= \int_a^b M(x)\,dx \end{aligned}$$

as we saw in (9).

Flow Integrals and Circulation

If the vector field

$$\mathbf{F} = M(x, y, z)\,\mathbf{i} \;+\; N(x, y, z)\,\mathbf{j} \;+\; P(x, y, z)\,\mathbf{k}$$

represents the velocity field of a fluid flowing through a region in space then the integral of $\mathbf{F} \cdot \mathbf{T}$ along a smooth curve in the region gives the fluid's flow along the curve. In that case, we have the following

Definition.

If C is a smooth curve in the domain of a continuous velocity field

$$\mathbf{F} = M(x, y, z)\,\mathbf{i} \;+\; N(x, y, z)\,\mathbf{j} \;+\; P(x, y, z)\,\mathbf{k},$$

then the **flow** along the curve from $t = a$ to $t = b$ is

(10) $$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

where $\mathbf{T}$ is the unit tangent vector. This is called the **flow integral**. If the curve is a closed loop, the flow is called the **circulation** around the curve.

Example 5. Flow Integral

Let $\mathbf{F} = -4xy\,\mathbf{i} + 8y\,\mathbf{j} + 2\,\mathbf{k}$ be a velocity field. Find the flow along the curve $C : \mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} +\ \mathbf{k},\ 0 \le t \le 2$.

Observe that

$$d\mathbf{r} = (\,\mathbf{i} + 2t\,\mathbf{j})\,dt$$
$$\mathbf{F}(\mathbf{r}(t)) = -4(t)(t^2)\,\mathbf{i} + 8t^2\,\mathbf{j} + 2\,\mathbf{k}$$

So by (10) we must evaluate

$$\int_C \mathbf{F} \cdot \mathbf{T}\,ds = \int_{t=0}^{t=2} \mathbf{F} \cdot d\mathbf{r}$$

Here the right-hand side is one of the equivalent forms listed on page 8. Continuing we have

$$\begin{aligned}\int_{t=0}^{t=2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \left(-4t^3\,\mathbf{i} + 8t^2\,\mathbf{j} + 2\,\mathbf{k}\right) \cdot (\,\mathbf{i} + 2t\,\mathbf{j})\,dt \\ &= \int_0^2 \left(-4t^3 + 16t^3\right)\,dt \\ &= 12\int_0^2 t^3\,dt = 48\end{aligned}$$

Example 6. Circulation

Let $\mathbf{F} = y\,\mathbf{i} + 2xy\,\mathbf{j}$ be a velocity field. Find the counter-clockwise circulation around the upper half of the unit circle. So let

$$C_1 : \mathbf{r}_1(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}, \quad 0 \le t \le \pi$$
$$C_2 : \mathbf{r}_2(t) = t\,\mathbf{i}, \quad -1 \le t \le 1$$

It follows that

$$d\mathbf{r}_1 = (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$
$$\mathbf{F}(\mathbf{r}_1(t)) = \sin t\,\mathbf{i} + 2\cos t \sin t\,\mathbf{j}$$

Also,

$$d\mathbf{r}_2 = \mathbf{i}\,dt \quad \text{and} \quad \mathbf{F}(\mathbf{r}_2(t)) = \mathbf{0}$$

It follows that the circulation is given by

$$
\begin{aligned}
\int_{C_1 \cup C_2} \mathbf{F} \cdot \mathbf{T}\, ds &= \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds + \int_{C_2} \mathbf{F} \cdot \mathbf{T}\, ds \\
&= \int_{C_1} \mathbf{F} \cdot \mathbf{T}\, ds + 0 \\
&= \int_{t=0}^{t=\pi} \mathbf{F} \cdot d\mathbf{r}_1 \\
&= \int_0^{\pi} \left(2\cos^2 t \sin t - \sin^2 t\right)\, dt \\
&= 2\int_0^{\pi} \cos^2 t \sin t\, dt - \int_0^{\pi} \sin^2 t\, dt \\
&= -2\int_1^{-1} u^2\, du - \frac{1}{2}\int_0^{\pi} (1 - \cos 2t)\, dt \\
&= \vdots \\
&= \frac{4}{3} - \frac{\pi}{2}
\end{aligned}
$$

Remark. In section 16.4 we will discover another way to evaluate the above integral.

Example 7. Evaluate the integral below around the closed curves that follow.

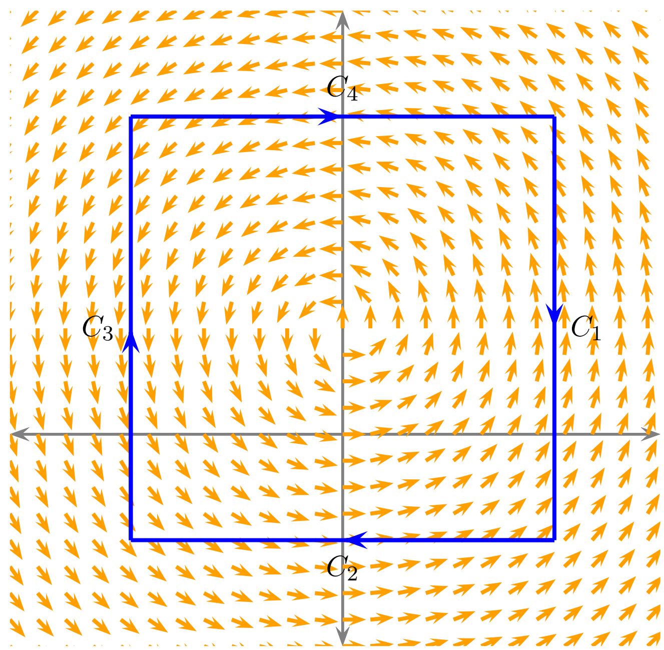
(11) $$\int_C \frac{1-y}{x^2+(y-1)^2}\,dx + \frac{x}{x^2+(y-1)^2}\,dy$$

(a) C is the rectangle with corners $(2,3),(2,-3),(-2,-3),(-2,3)$ and the curve is traversed clockwise (once) when viewed from above.

So let
$$\mathbf{F} = \frac{1-y}{x^2+(y-1)^2}\,\mathbf{i} + \frac{x}{x^2+(y-1)^2}\,\mathbf{j}$$

be a velocity field. Then the integral in (11) can be viewed as the (clockwise) circulation integral $\oint_C \mathbf{F}\cdot d\mathbf{r}$.

In class we showed that
$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = -\tan^{-1}2 - \tan^{-1}1$$
Similar calculations show that
$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = -\tan^{-1}2 - \tan^{-1}1$$
and
$$\int_{C_4} \mathbf{F}\cdot d\mathbf{r} = -2\tan^{-1}1$$



Not to scale

We evaluate $\int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2$ below.

Notice that C_2 can be parameterized by the vector equation $\mathbf{r}_2(t) = (2-4t)\,\mathbf{i} - 3\,\mathbf{j},\ 0 \le t \le 1$. It follows that $d\mathbf{r}_2 = -4\,\mathbf{i}\,dt$, and

$$\mathbf{F}(\mathbf{r}_2(t)) = \frac{1}{(1-2t)^2+4}\,\mathbf{i} + \frac{2-4t}{(1-2t)^2+4}\,\mathbf{j}$$

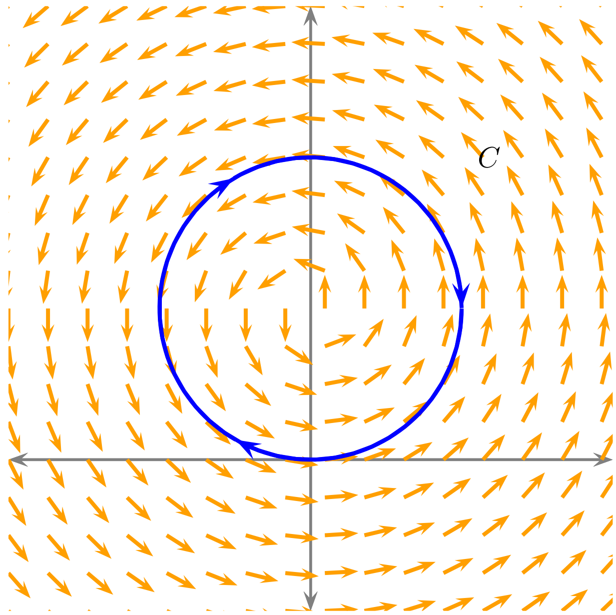
so that

$$\begin{aligned}
\int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 &= \int_0^1 \frac{-4}{(1-2t)^2+4}\,dt \\
&= \int_1^{-1} \frac{2}{u^2+4}\,du \\
&= \frac{1}{2}\int_1^{-1} \frac{1}{(u/2)^2+1}\,du \\
&= \tan^{-1}(u/2)\Big|_1^{-1} \\
&= -2\tan^{-1}(1/2)
\end{aligned}$$

Now let I denote the integral in (11). Putting this all together yields

$$\begin{aligned}
I &= \oint_C \mathbf{F}\cdot d\mathbf{r} \\
&= \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}_3 + \int_{C_4} \mathbf{F}\cdot d\mathbf{r}_4 \\
&= -2(\tan^{-1} 2 + \tan^{-1} 1 + \tan^{-1}(1/2) + \tan^{-1} 1) \\
&= -2\pi
\end{aligned}$$

(b) C is the circle of radius 1 centered at $(0, 1)$ traversed clockwise.



Now C can be parameterized by the vector equation $\mathbf{r}(t) = \cos t\ \mathbf{i} + (1 - \sin t)\ \mathbf{j},\ 0 \le t \le 2\pi$ and

$$\mathbf{F}(\mathbf{r}(t)) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j}$$

and

$$d\mathbf{r} = -(\sin t\ \mathbf{i} + \cos t\ \mathbf{j})\, dt$$

so that

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} -(\sin^2 t + \cos^2 t)\, dt \\ &= -2\pi\end{aligned}$$

We will have more to say about this example in section 16.4.