15.3 Double Integrals over General Regions

Theorem 1. Fubini's Theorem (Stronger Form)

If $f(x,y)$ is continuous over a region R.

1. If R is defined by $a \le x \le b,\ g_1(x) \le y \le g_2(x)$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_{y=g_1(x)}^{y=g_2(x)} f(x,y)\,dy\,dx$$

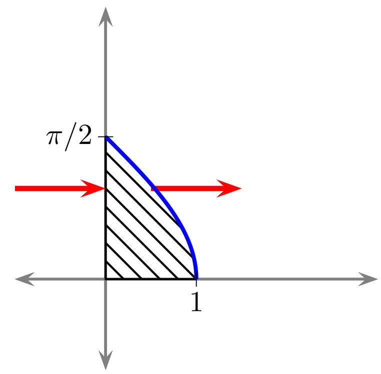
2. If R is defined by $c \le y \le d,\ h_1(y) \le x \le h_2(y)$, then

$$\iint_R f(x,y)\,dA = \int_c^d \int_{x=h_1(y)}^{x=h_2(y)} f(x,y)\,dx\,dy$$

Example 1. Consider the double integral below and answer the questions that follow.

$$\int_0^1 \int_0^{\cos^{-1} x} e^{\sin y}\, dy\, dx$$

(a) Sketch the region of integration.



(b) Write an equivalent double integral with the order of integration reversed.

From the sketch. Notice that the red arrows indicate the new direction for the inside integral.

$$\int_0^{\pi/2} \int_{x=0}^{x=\cos y} e^{\sin y}\, dx\, dy$$

(c) Evaluate the double integral above.

Since the above integrals are equal, we will use the one from part (b) (why?).

$$= \int_0^{\pi/2} e^{\sin y} \cos y\, dy$$

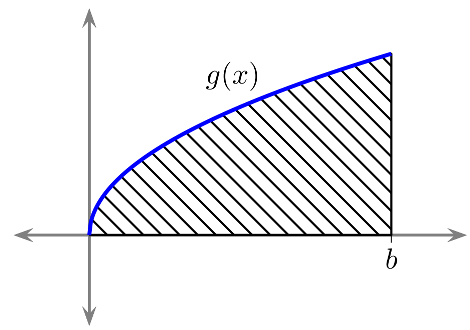
$$= e^{\sin y} \Big|_0^{\pi/2} = e - 1$$

Example 2. Let h be an integrable function over the region R defined by

$$R\colon\ 0 \le y \le g(x), \quad 0 \le x \le b$$

Here $g(x)$ is increasing on $(0, b)$. See the sketch below.

(1) $$\iint_R h(x,y)\,dA = \int_0^b \int_0^{g(x)} h(x,y)\,dy\,dx$$



Rewrite the iterated integral in (1) with the order of integration reversed.

Notice that $g^{-1}(y)$ is defined on the interval $(0, g(b))$. Thus

$$\iint_R h(x,y)\,dA = \int_0^{g(b)} \int_{g^{-1}(y)}^{b} h(x,y)\,dx\,dy$$

Exercise - Continuing with the above example, suppose that h is an integrable function defined over the region $S\colon\ g(x) \le y \le g(b),\ 0 \le x \le b$. Rewrite the iterated integral below with the order of integration reversed.

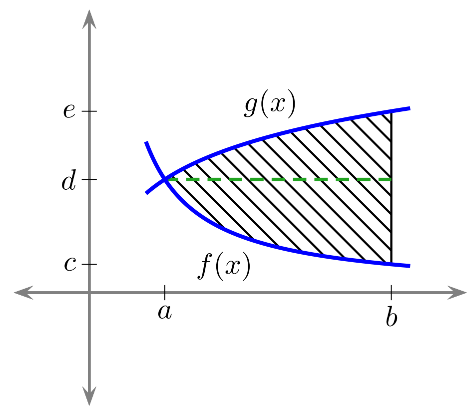
$$\iint_S h(x,y)\,dA = \int_0^b \int_{g(x)}^{g(b)} h(x,y)\,dy\,dx$$

Example 3. Let h be an integrable function over the region R where

$$R\colon\ f(x) \le y \le g(x), \quad a \le x \le b$$

Here $f(x)$ is decreasing on (a,b) and $g(x)$ is increasing on (a,b). See the sketch below. Now

(2) $$\iint_R h(x,y)\,dA = \int_a^b \int_{f(x)}^{g(x)} h(x,y)\,dy\,dx$$



Rewrite the iterated integral with the order of integration reversed (as unrealistic as that might be in this case).

Let $c = f(b),\ d = f(a) = g(a),$ and $e = g(b)$ (see the sketch). Notice that $f^{-1}(y)$ is defined on the interval (c,d) and $g^{-1}(y)$ is defined on the interval (d,e). Now let I denote the integral in (2). Then $I = I_1 + I_2$ where

$$I_1 = \int_c^d \int_{f^{-1}(y)}^b h(x,y)\,dx\,dy$$

and

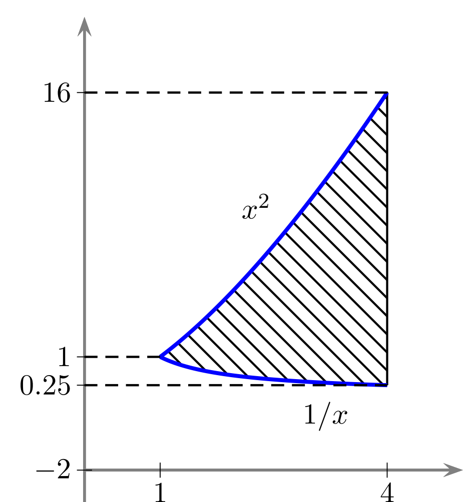
$$I_2 = \int_d^e \int_{g^{-1}(y)}^b h(x,y)\,dx\,dy$$

Example 4. Confirm each of the calculations below by evaluating each integral in two different ways.

a. $\displaystyle\int_0^4 \int_0^{\sqrt{x}} 2x^2 y \, dy \, dx = 64$

b. $\displaystyle\int_1^4 \int_{1/x}^{x^2} 8xy \, dy \, dx = 2730 - 4 \ln 4$

The region of integration is shown in the figure to the right.



Area of Bounded Regions in the Plane

Let $f(x)$ be a nonnegative function defined on the interval $[a, b]$. In a first semester calculus course we found the “area under a curve $y = f(x)$ between a and b” was given by

$$\text{Area} = \int_a^b f(x)\,dx$$

The following definition extends this notion.

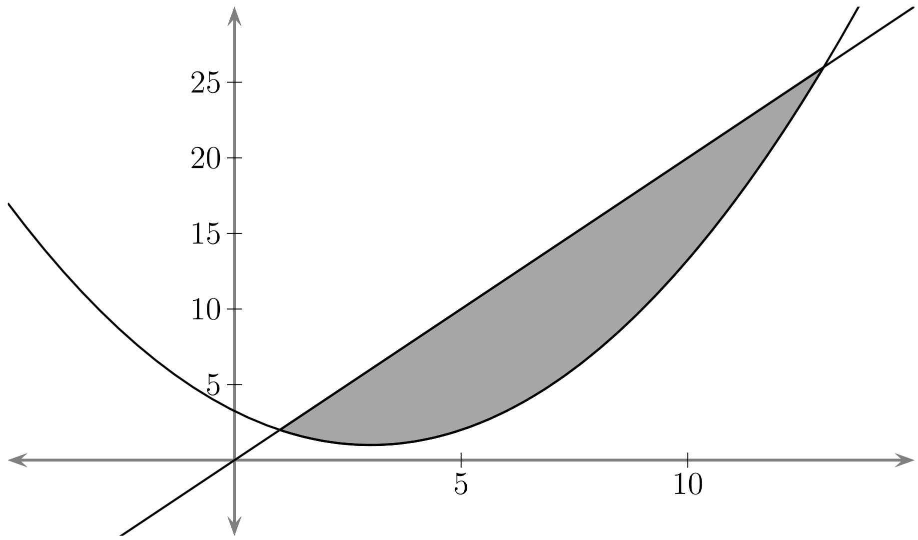
Definition. Area

The **area** of a closed, bounded region R in the plane is

$$A = \iint_R dA$$

Example 5. Finding the area of a planar region.

Find the area of the region R bounded by the curves $y = (x-3)^2/4 + 1$ and $y = 2x$. (See sketch.)



$$\text{Area} = \iint_R dA$$

$$= \int_1^{13} \int_{(x-3)^2/4+1}^{2x} dy\,dx$$

Average Value of a Function

As we did in Calculus I, we have the following definition.

Definition. Let f be an integrable function defined over a region R in the plane. Then the **average value of f over R** is given by

$$f_{\text{avg}} = \frac{1}{\text{area of } R} \iint_R f\, dA$$

Example 6. Finding the average value of a function.

Let $f(x, y) = 2x$. Find the average value of f over the region R from the example above. We have

$$\begin{aligned} f_{\text{avg}} &= \frac{1}{72} \iint_R f\, dA \\ &= \frac{1}{72} \int_1^{13} \int_{(x-3)^2/4+1}^{2x} 2x\, dy\, dx \end{aligned}$$

Mass and moment formulas for thin plates covering regions in the xy-plane.

Density: $\delta(x, y)$ (mass per unit area)

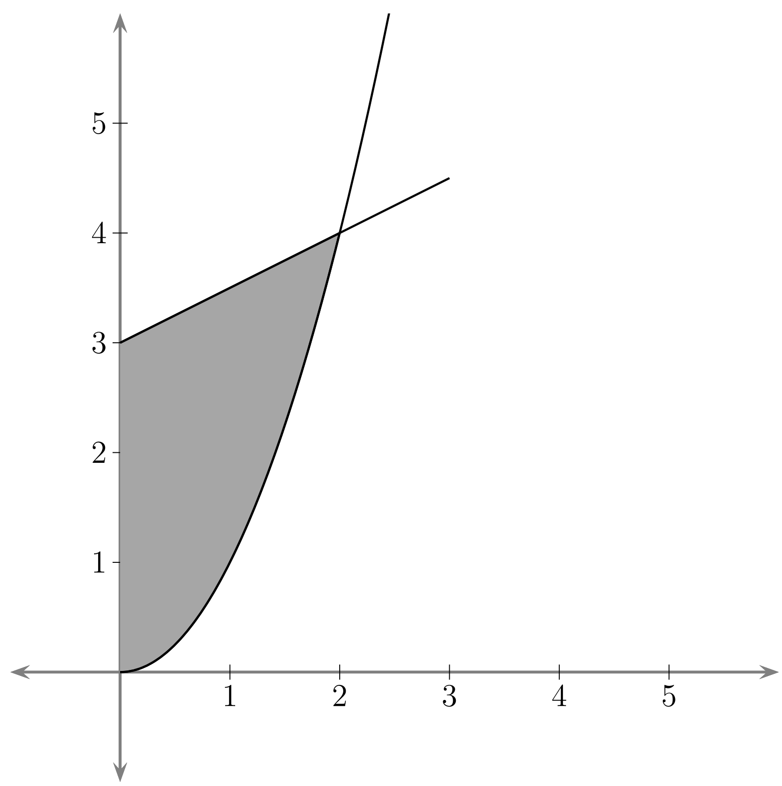
Mass: $M = \iint \delta(x, y)\, dA$

First Moments: $M_x = \iint y\, \delta(x, y)\, dA$ and $M_y = \iint x\, \delta(x, y)\, dA$

Center of Mass: $\overline{x} = \dfrac{M_y}{M},\ \overline{y} = \dfrac{M_x}{M}$

Example 7. Finding the center of mass.

Let R be the region (see sketch) defined by $0 \le x \le 2, \;\; x^2 \le y \le \dfrac{x}{2} + 3$ with a density function $\delta(x, y) = 1$. Find the center of mass.



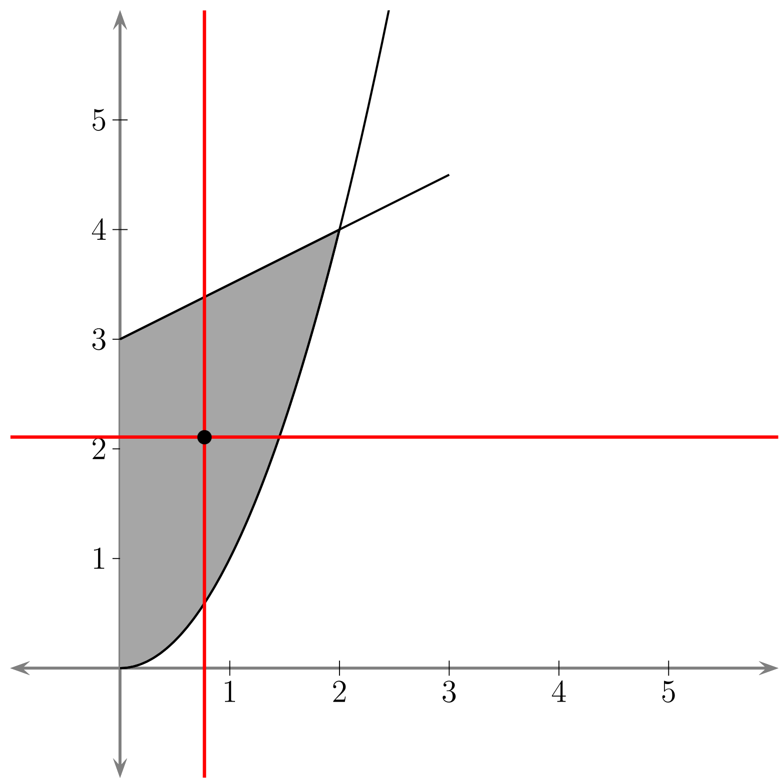
$$
\begin{aligned}
M &= \iint_R \delta(x,y)\,dA \\
&= \int_0^2 \int_{x^2}^{x/2+3} dy\,dx \\
&= \int_0^2 \left(\frac{x}{2} + 3 - x^2\right)\,dx \\
&= \frac{13}{3}
\end{aligned}
$$

$$
\begin{aligned}
M_x &= \iint_R y\,\delta(x,y)\,dA \\
&= \int_0^2 \int_{x^2}^{x/2+3} y\,dy\,dx \\
&= \frac{1}{2}\int_0^2 \left(y^2 \Big|_{x^2}^{x/2+3} \right)\,dx \\
&= \frac{1}{8}\int_0^2 \left(-4x^4 + x^2 + 12x + 36\right)\,dx \\
&= \frac{137}{15}
\end{aligned}
$$

$$
\begin{aligned}
M_y &= \iint_R x\,\delta(x,y)\,dA \\
&= \int_0^2 \int_{x^2}^{x/2+3} x\,dy\,dx \\
&= \int_0^2 \left(xy \,\Bigg|_{y=x^2}^{y=x/2+3} \right) dx \\
&= \frac{1}{2}\int_0^2 (x^2 + 6x - 2x^3)\,dx \\
&= \frac{10}{3}
\end{aligned}
$$

$$
\begin{aligned}
\overline{y} &= \frac{M_x}{M} = \frac{137}{65} \\
\overline{x} &= \frac{M_y}{M} = \frac{10}{13}
\end{aligned}
$$

It follows that the center of mass is $\left(\frac{10}{13}, \frac{137}{65}\right)$ as indicated below.



In the example above the density function was constant. In these cases the mass is said to be “**uniformly distributed**” or **homogeneous**.