15.7 Triple Integrals in Rectangular Coordinates

Triple Integrals

Suppose that $f(x, y, z)$ is defined on a closed bounded region D in space. Can we define the integral of f over D? Proceeding in the usual way (that is, partitioning the region D, etc.), we obtain the following (Riemann) sum

$$S_n = \sum_{k=1}^{n} f\left(x_k, y_k, z_k\right) \triangle V_k$$

where $\triangle V_k = \triangle x_k \triangle y_k \triangle z_k$.

Now we take the limit of the above expression as $\|P\| \to 0$, where $\|P\|$ is the norm of the partition P. If the limit exists we say that f **is integrable over** D and write

$$\iiint_D f(x, y, z)\, dV = \lim_{\|P\| \to 0} \sum_{k=1}^{n} f\left(x_k, y_k, z_k\right) \triangle V_k$$

It turns out that if f is continuous over the closed bounded region D then f is integrable (as long as D is “reasonable”). Also, the above integral can actually be computed using an iterated integral as we did in the two-dimensional case.

The volume of a region in space.

Definition. Volume

The **volume** of a closed bounded region in space is

$$V = \iiint_D dV$$

Theorem 1. Properties of Triple Integrals

If $F = F(x, y, z)$ and $G = G(x, y, z)$ are continuous, then

1. $\iiint_D k\,F\,dV = k\iiint_D F\,dV, \qquad k \in \mathbb{R}$

2. $\iiint_D (F \pm G)\;dV = \iiint_D F\,dV \pm \iiint_D G\,dV$

3. $F \geq 0 \Longrightarrow \iiint_D F\,dV \geq 0$

4. $F \geq G$ on $D \Longrightarrow \iiint_D F\,dV \geq \iiint_D G\,dV$

5. If D is the union of nonoverlapping cells $D_1,\ D_2,\ \ldots, D_n$ then

$$\iiint_D F\,dV = \iiint_{D_1} F\,dV + \iiint_{D_2} F\,dV + \cdots + \iiint_{D_n} F\,dV$$

Example 1.

Evaluate the following integrals

a. $\int_0^3 \int_0^2 \int_0^{x^2+3y^2} dz\, dy\, dx$

$$
\begin{aligned}
&= \int_0^3 \int_0^2 (x^2 + 3y^2)\, dy\, dx \\
&= \int_0^3 (x^2 y + y^3) \Big|_{y=0}^{y=2} dx \\
&= \int_0^3 (2x^2 + 8)\, dx \\
&= \left(\frac{2x^3}{3} + 8x \right) \Big|_0^3 \\
&= 18 + 24
\end{aligned}
$$

Notice that this just the volume of the region between the xy-plane and the surface $z = f(x, y) = x^2 + 3y^2$ over the rectangle $[0, 3] \times [0, 2]$.

b. $$\int_0^{\sqrt{2}}\int_0^{\sqrt{4-2y^2}}\int_{x^2+3y^2}^{8-x^2-y^2} dz\,dx\,dy = I$$

$$\begin{aligned}
&= \int_0^{\sqrt{2}}\int_0^{\sqrt{4-2y^2}} \left[\left(8-x^2-y^2\right)-\left(x^2+3y^2\right)\right]\,dx\,dy\\
&= \int_0^{\sqrt{2}}\int_0^{\sqrt{4-2y^2}} \left[8-2x^2-4y^2\right]\,dx\,dy\\
&= \int_0^{\sqrt{2}} \left[8x-\frac{2x^3}{3}-4y^2x\right]\Big|_{x=0}^{x=\sqrt{4-2y^2}}\,dy\\
&= \int_0^{\sqrt{2}} \left[8\sqrt{4-2y^2}-\frac{2}{3}(4-2y^2)^{3/2}-4y^2\sqrt{4-2y^2}\right]\,dy\\
&= I_1+I_2+I_3
\end{aligned}$$

Now let $\sqrt{2}y = 2\sin\theta$. Then $dy = \sqrt{2}\cos\theta\,d\theta$, etc. and

$$\begin{aligned}
I_1 &= 8\int_0^{\sqrt{2}} \sqrt{4-2y^2}\,dy\\
&= 16\sqrt{2}\int_0^{\pi/2}\cos^2\theta\,d\theta\\
&= 8\sqrt{2}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta\\
&= 4\sqrt{2}\pi
\end{aligned}$$

We leave it as an exercise to confirm that

$$I_2 = \frac{-2}{3}\int_0^{\sqrt{2}}(4-2y^2)^{3/2}\,dy = -\sqrt{2}\pi$$

$$I_3 = -4\int_0^{\sqrt{2}} y^2\sqrt{4-2y^2}\,dy = -\sqrt{2}\pi$$

It follows that

$$\begin{aligned}
I &= I_1+I_2+I_3\\
&= 2\sqrt{2}\pi
\end{aligned}$$

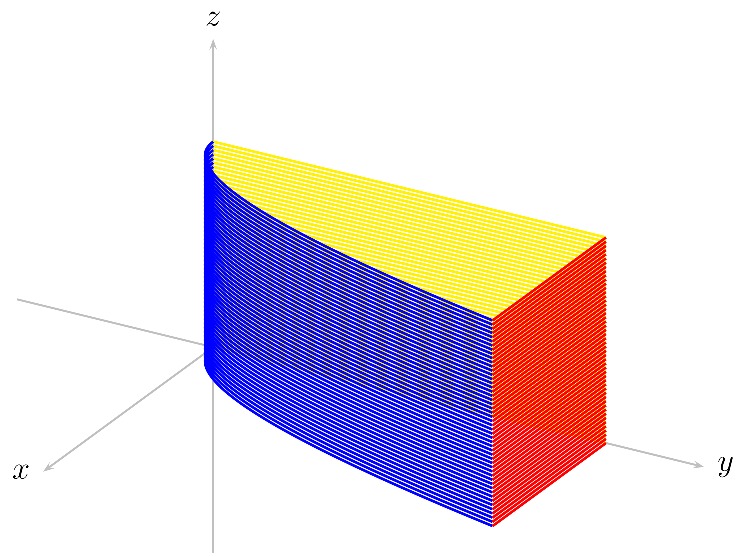


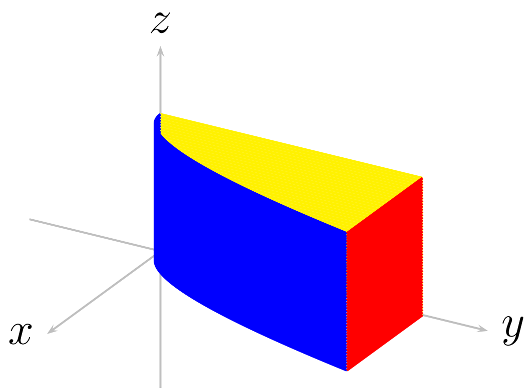
Figure 1: $T(x, y, z) = 12xz\, e^{zy^2}$ defined over a region in space

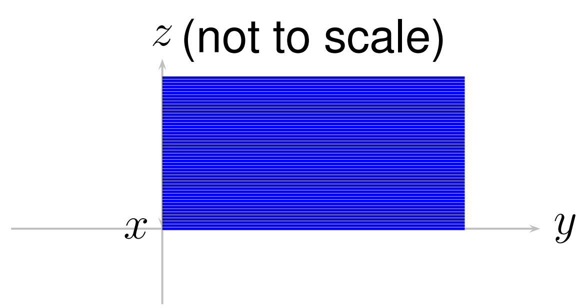
c. Let $T(x, y, z) = 12xz\, e^{zy^2}$. Evaluate the integral below.

$$\int_0^1 \int_0^1 \int_{x^2}^1 T(x, y, z)\, dy\, dx\, dz$$

Notice that the integrand has no elementary antiderivative. Perhaps a change in the order of integration might help, as we saw with double integrals. We try to integrate first with respect to x.

It follows that the limits of integration of the inner-most integral are from $x = 0$ to $x = \sqrt{y}$. What about the remaining limits? Once we complete the integration in the x-direction, we project the solid onto the remaining coordinate system. In this case, that means we project the solid onto the yz-plane to obtain the sketch below (on the right).





Notice that we end up with the one by one square $[0,1] \times [0,1]$. Thus

$$\begin{aligned}\int_0^1 \int_0^1 \int_{x^2}^1 12xz\, e^{zy^2}\, dy\, dx\, dz &= \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xz\, e^{zy^2}\, dx\, dy\, dz \\ &= \int_0^1 \int_0^1 12z\, e^{zy^2} \int_0^{\sqrt{y}} x\, dx\, dy\, dz \\ &= \int_0^1 \int_0^1 6z\, e^{zy^2} \left[(\sqrt{y})^2 - 0\right]\, dy\, dz \\ &= \int_0^1 \int_0^1 6yz\, e^{zy^2}\, dy\, dz = I\end{aligned}$$

Now let $w(y) = zy^2$. Then $dw = 2zy\,dy$, $w(0) = 0$ and $w(1) = z$. Then

$$
\begin{aligned}
I &= \int_0^1 \int_0^z 3\,e^w\,dw\,dz \\
&= 3\int_0^1 e^w \,\Big|_0^z\, dz \\
&= 3\int_0^1 \left(e^z - 1\right)\,dz \\
&= 3\left(e^z - z\right)\,\Big|_0^1 = 3(e-2)
\end{aligned}
$$

How might we interpret this result?

Suppose that the integrand, $T(x,y,z) = 12xz\,e^{zy^2}$ gave the temperature over the region D shown in Figure 1. An easy calculation shows that the volume of D is

$$V = \int_0^1 \int_0^1 \int_{x^2}^1 dy\,dx\,dz = \frac{2}{3}$$

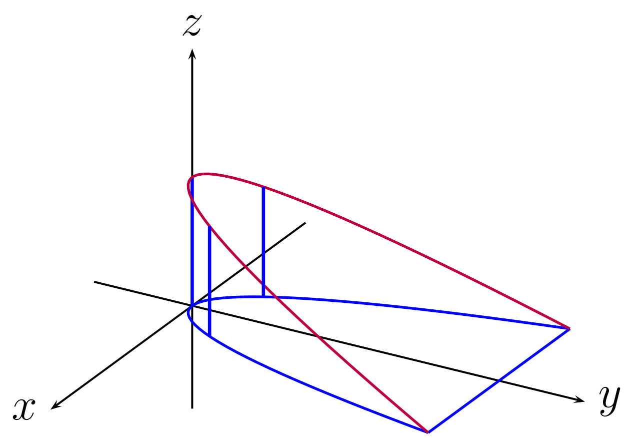
Then the average temperature over the region would be

$$
\begin{aligned}
T_{\text{avg}} &= \frac{1}{V}\int_0^1 \int_0^1 \int_{x^2}^1 T(x,y,z)\,dy\,dx\,dz \\
&= \frac{1}{2/3} 3(e-2) \approx 3.232
\end{aligned}
$$

Example 2. Volumes

The volume of a the solid shown is given by the triple integral

$$\int_{-2.5}^{2.5}\int_{x^2}^{6.25}\int_{0}^{(6.25-y)/2.5} dz\, dy\, dx$$



Find the volume by evaluating the iterated integral above.

Here and below we let $c = 2.5$.

$$V = \int_{-c}^{c} \int_{x^2}^{c^2} \int_{0}^{(c^2-y)/c} dz\, dy\, dx$$

$$= \frac{1}{c} \int_{-c}^{c} \int_{x^2}^{c^2} (c^2 - y)\, dy\, dx$$

$$= \frac{1}{c} \int_{-c}^{c} \left(c^2 y - \frac{y^2}{2} \right) \Big|_{x^2}^{c^2} dx$$

$$= \frac{1}{c} \int_{-c}^{c} \left[\left(c^4 - \frac{c^4}{2} \right) - \left(c^2 x^2 - \frac{x^4}{2} \right) \right] dx$$

and since the integrand is even

$$= \frac{2}{c} \int_{0}^{c} \left(\frac{c^4}{2} - c^2 x^2 + \frac{x^4}{2} \right) dx$$

(1)
$$= \frac{1}{c} \int_{0}^{c} \left(c^4 - 2c^2 x^2 + x^4 \right) dx$$

$$= \frac{2}{c} \left(\frac{c^4 x}{2} - \frac{c^2 x^3}{3} + \frac{x^5}{10} \right) \Big|_{0}^{c}$$

$$= \frac{2}{c} \left(\frac{c^5}{2} - \frac{c^5}{3} + \frac{c^5}{10} \right)$$

$$= \frac{125}{6}$$

Example 3. Changing the Order of Integration

Now rewrite the integral from the last example by changing the order of integration using each of the other 5 possibilities.

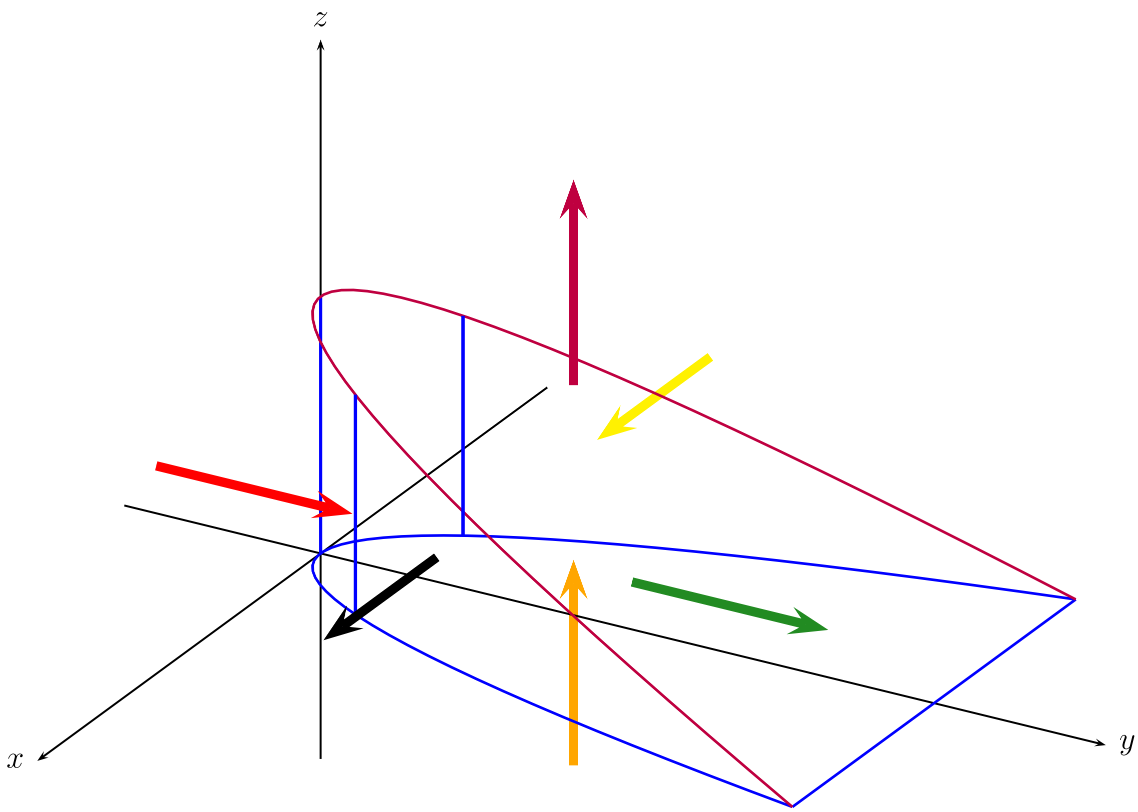
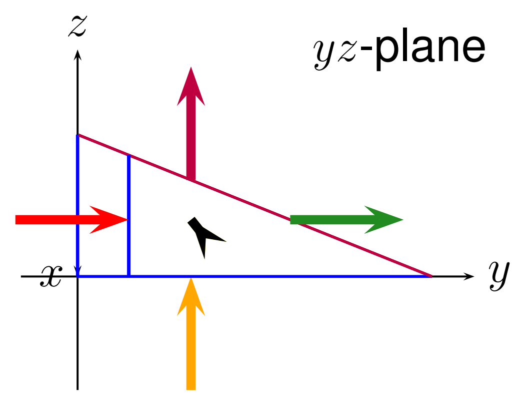
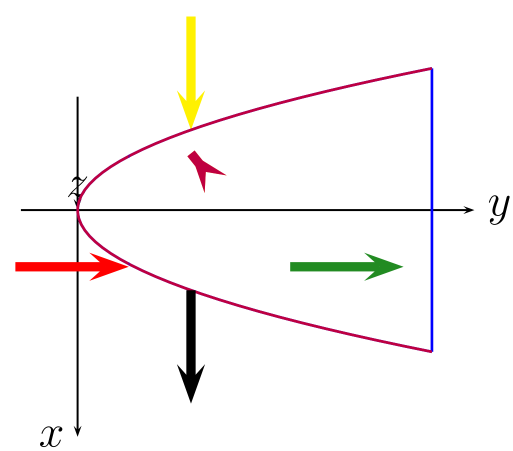
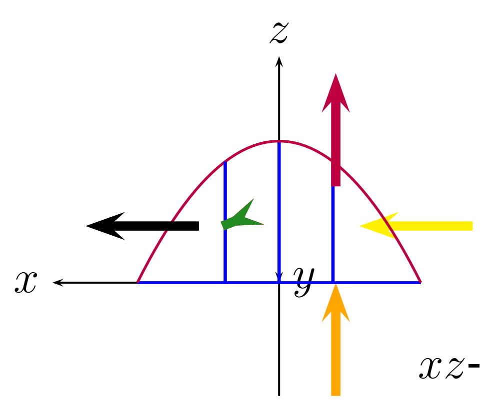


Figure 2: Changing the Order of Integration

First consider the following profiles.







(a) We first try the order $dy\,dz\,dx$. We use the sketches above to help us determine the new limits of integration.

(i) We first determine the new limits for y (as a function of x and z).

Notice that the entry surface (red arrow) is parallel to the z-axis and hence independent of z. It follows that $y = x^2$.

The exit surface (green arrow) is parallel to the x-axis and hence independent of x. Thus we solve the equation below for $y = f(z)$.

$$\frac{y}{6.25} + \frac{z}{2.5} = 1$$

It follows that

$$\begin{aligned} y &= f(z) \\ &= 6.25\left(1 - \frac{z}{2.5}\right) \\ &= c^2 - cz \end{aligned}$$

and $x^2 \leq y \leq c^2 - cz$.

Notice that this information must be gathered from the main sketch.

(ii) Now z ranges from 0 to some unknown function $z = h(x)$ (see the xy-plane view). To find $h(x)$, let $P = P(x, y, z)$ be a point on the boundary of the upper surface of the shoehorn in Figure 2. Then the corresponding point on the curve $z = h(x)$ must be $Q = Q(x, 0, z)$. Now a careful inspection reveals that

$$\begin{aligned} P &= P(x, y, z) \\ &= P(x, y, (c^2 - y)/c) \\ &= P(x, x^2, (c^2 - x^2)/c) \end{aligned}$$

It follows that

$$Q = Q(x, 0, (c^2 - x^2)/c)$$

In other words

$$z = \frac{c^2 - x^2}{c}$$

So that

$$0 \le z \le \frac{c^2 - x^2}{c}$$

(iii) Finally, we use the xz-plane to project onto the x-axis to determine

$$-c \le x \le c$$

It follows that

$$
\begin{aligned}
V &= \int_{-c}^{c} \int_{0}^{(c^2-x^2)/c} \int_{x^2}^{c^2-cz} dy\, dz\, dx \\
&= \int_{-c}^{c} \int_{0}^{(c^2-x^2)/c} (c^2 - x^2 - cz)\, dz\, dx \\
&= \int_{-c}^{c} \left[\left(c^2 - x^2\right) z - \frac{cz^2}{2} \right] \Big|_{0}^{(c^2-x^2)/c} dx \\
&= \frac{1}{2c} \int_{-c}^{c} \left(c^2 - x^2\right)^2 \, dx \\
&= \frac{1}{c} \int_{0}^{c} \left(x^4 - 2c^2x^2 + c^4\right) \, dx
\end{aligned}
$$

which is (1). Thus

$$
V = \frac{125}{6}
$$

as we saw above.

(b) Next we try $dy\,dx\,dz$.

(i) Same as above:

$$x^2 \leq y \leq c^2 - cz$$

(ii) Now we use the xz-plane view to determine the limits with respect to the x-axis.

Notice that the entry point is through the curve $x = -\sqrt{c^2 - cz}$ and similarly the exit point is through the curve $x = \sqrt{c^2 - cz}$ hence

$$-\sqrt{c^2 - cz} \leq x \leq \sqrt{c^2 - cz}$$

(iii) Finally, we use xz-plane to project onto the z-axis to determine

$$0 \leq z \leq c$$

Now let $u(z) = \sqrt{c^2 - cz}$. Then

$$
\begin{aligned}
V &= \int_0^c \int_{-u(z)}^{u(z)} \int_{x^2}^{u(z)^2} dy\, dx\, dz \\
&= \int_0^c \int_{-u(z)}^{u(z)} \left(u(z)^2 - x^2\right)\, dx\, dz \\
&= \int_0^c \left(u(z)^2 x - \frac{x^3}{3}\right) \Big|_{-u(z)}^{u(z)} dz \\
&= \frac{4}{3} \int_0^c u(z)^3\, dz \\
&= \frac{4}{3} \int_0^c \left(c^2 - cz\right)^{3/2}\, dz \\
&= \frac{-8}{15c} \left(c^2 - cz\right)^{5/2}\, \Big|_0^c \\
&= \vdots \\
&= \frac{125}{6}
\end{aligned}
$$

as we saw above.

(c) Notice that it is easier to reverse the order of the inner two or outer two limits of integration. Continuing along these lines we try $dx\,dy\,dz$.

(i) Returning to the original sketch, we see that $x = k(y)$ since the entry and exit surfaces are parallel to the z-axis. In fact,

$$-\sqrt{y} \leq x \leq \sqrt{y}$$

(ii) From the yz-plane we see that

$$0 \leq y \leq c^2 - cz$$

(iii) Finally,

$$0 \leq z \leq c$$

Thus

$$V = \int_0^c \int_0^{c^2-cz} \int_{-\sqrt{y}}^{\sqrt{y}} dx\,dy\,dz$$

Can you evaluate this iterated integral.

(d) As an exercise set up the corresponding iterated integrals for the differentials $dx\,dz\,dy$ and $dz\,dx\,dy$.