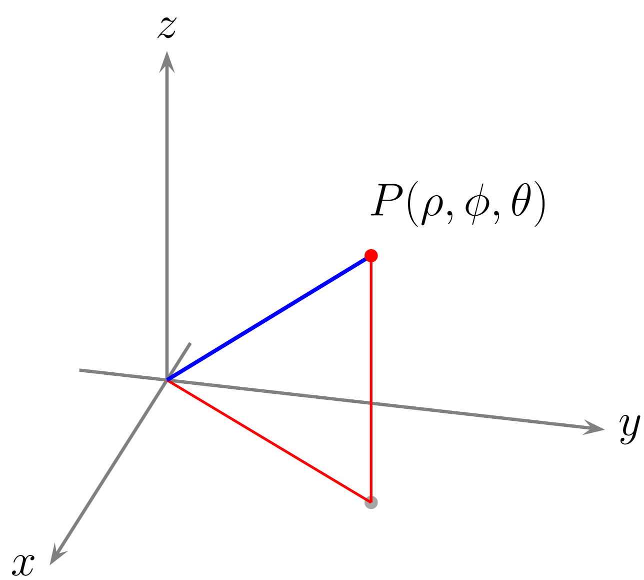
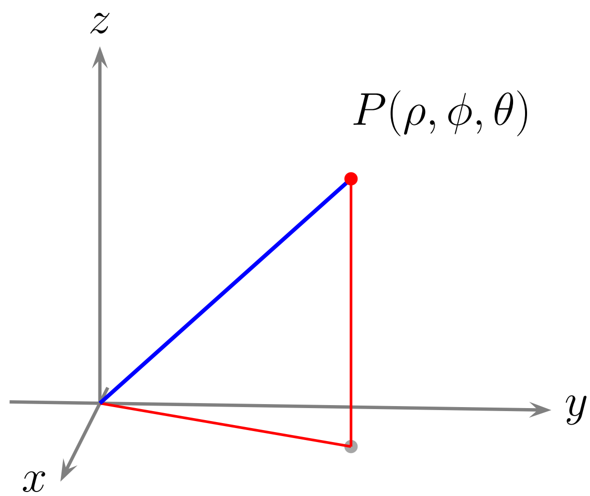
Integration and Spherical Coordinates

Definition. **Spherical coordinates** represent a point P in space by the ordered triple (ρ, ϕ, θ) where

1. ρ is the **distance** from P to the origin.
2. ϕ is the angle that $\overrightarrow{OP}$ makes with the positive z-axis $(0 \leq \phi \leq \pi)$.
3. θ is the angle from cylindrical coordinates.

Remark. In particular, $\rho \geq 0$.





The following equations relate spherical coordinates to rectangular and cylindrical coordinates.

$$r = \rho \sin\phi, \quad x = r\cos\theta = \rho\sin\phi\cos\theta,$$

$$z = \rho\cos\phi, \quad y = r\sin\theta = \rho\sin\phi\sin\theta,$$

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2}$$

Example 1. Constant-Coordinates Equations

Describe the objects generated by the constant equations:

$$\rho = \rho_0$$

$$\phi = \phi_0$$

$$\theta = \theta_0$$

Example 2. Converting Spherical Coordinates

Consider the spherical equation $\phi = \dfrac{\pi}{3}$. Find the equivalent cylindrical and rectangular equations.

1. **First Attempt:**

   (a) **Cylindrical Coordinate Equation:** We've already looked at the cross-sections $z =$ const (≥ 0). Notice that the if $y = 0$ we must have the equation $\tan\phi = x/z$. Thus

   $$\frac{x}{z} = \tan\phi = \sqrt{3}$$
   $$\Longrightarrow x = \sqrt{3}\,z$$
   $$\Longrightarrow x^2 = 3z^2$$
   $$\Longrightarrow r^2 = 3z^2 \quad \text{(Why?)}$$

   (b) **Rectangular Coordinate Equation:** The last equation implies

   $$x^2 + y^2 = 3z^2$$

2. **Alternate Approach:**

$$\begin{aligned}\phi = \frac{\pi}{3} &\Longrightarrow \tan\phi = \sqrt{3} \\ &\Longrightarrow \frac{r}{z} = \sqrt{3} \\ &\Longrightarrow r^2 = 3z^2 \\ &\Longrightarrow \ldots\end{aligned}$$

Suppose that $f(\rho, \phi, \theta)$ is defined on a closed bounded region D in space. Can we define the integral of f over D? Proceeding as we did above (that is, partitioning the region D, etc.), we obtain the following (Riemann) sum

$$S_n = \sum_{k=1}^{n} f\left(\rho_k, \phi_k, \theta_k\right) \triangle V_k$$

where $\triangle V_k = \rho_k^2 \sin\phi_k \triangle\rho_k \triangle\phi_k \triangle\theta_k$.

Now we take the limit of the above expression as $\|P\| \to 0$, where $\|P\|$ is the norm of the partition P. If the limit exists we say that f **is integrable over** D and write

$$\begin{aligned}\lim_{n\to\infty} S_n &= \iiint_D f\,dV \\ &= \iiint_D f(\rho,\phi,\theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta\end{aligned}$$

Once again, if f is continuous over the closed bounded region D then f is integrable (as long as D is "reasonable").

Example 3. Integration - Spherical Coordinates

Evaluate the triple integral below.

$$I = \int_0^{2\pi} \int_0^{\pi/3} \int_{\cos\phi}^{2} 3\rho^2 \, \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\begin{aligned}
I &= 2\pi \int_0^{\pi/3} \rho^3 \, \sin\phi \, \Big|_{\rho=\cos\phi}^{\rho=2} d\phi \\
&= 2\pi \left(8 \int_0^{\pi/3} \sin\phi \, d\phi - \int_0^{\pi/3} \sin\phi \, \cos^3\phi \, d\phi \right) \\
&= 2\pi \left(-8 \, \cos\phi \, \Big|_0^{\pi/3} + \int_1^{1/2} u^3 \, du \right) \\
&= 2\pi \left(-8 \left(\frac{1}{2} - 1 \right) + \frac{1}{4} \left(\frac{1}{16} - 1 \right) \right) \\
&= 2\pi \left(4 - \frac{15}{64} \right)
\end{aligned}$$

Finding the limits of integration in spherical coordinates.

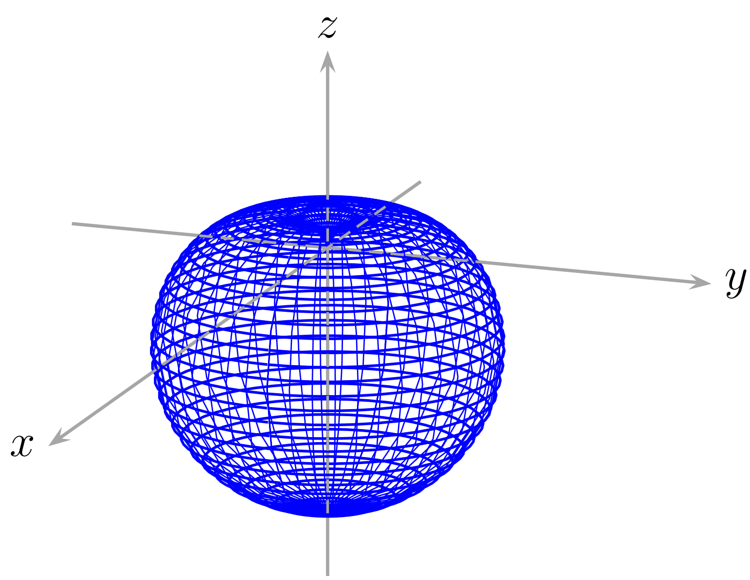
If $f(\rho, \phi, \theta)$ is continuous over a region $D \in \mathbb{R}^3$ then

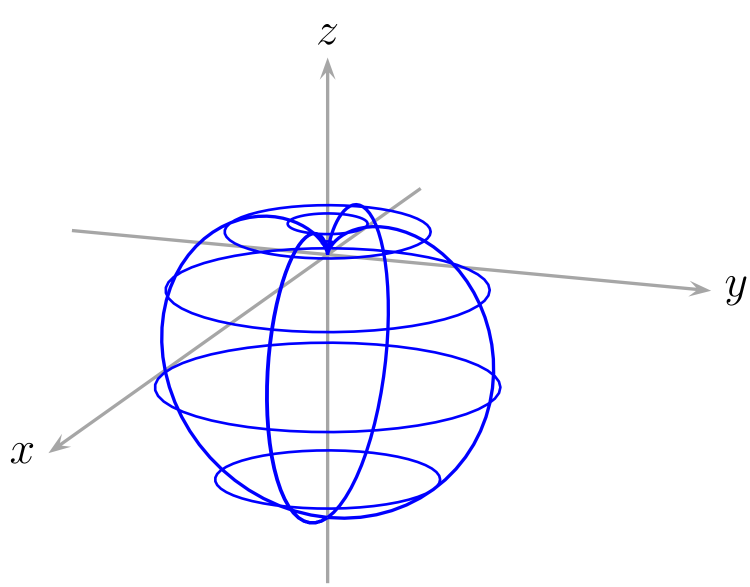
$$\iiint_D f\,dV = \iiint_D f(\rho, \phi, \theta)\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\text{min}}}^{\phi=\phi_{\text{max}}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta)\,d\rho\,d\phi\,d\theta$$

Example 4. Volume of a Cardioid of Revolution

Let D be the cardioid of revolution $\rho = 1 - \cos\phi$. Find the volume of D.

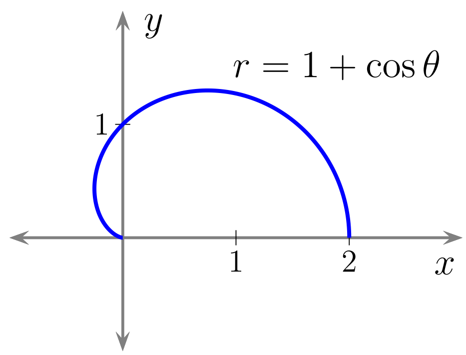




$$
\begin{aligned}
V &= \iiint_D \rho^2 \, \sin\phi \, d\rho \, d\phi \, d\theta \\
&= \int_{\theta=0}^{\theta=2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1-\cos\phi} \rho^2 \, \sin\phi \, d\rho \, d\phi \, d\theta \\
&= 2\pi \int_{\phi=0}^{\phi=\pi} \int_{\rho=0}^{\rho=1-\cos\phi} \rho^2 \, \sin\phi \, d\rho \, d\phi \\
&= \frac{2\pi}{3} \int_{\phi=0}^{\phi=\pi} \rho^3 \, \sin\phi \, \Big|_{\rho=0}^{\rho=1-\cos\phi} d\phi \\
&= \frac{2\pi}{3} \int_{\phi=0}^{\phi=\pi} \sin\phi \, (1-\cos\phi)^3 \, d\phi \\
&= \frac{2\pi}{3} \int_{u=0}^{u=2} u^3 \, du \\
&= \left(\frac{2\pi}{12}\right)(16-0) \\
&= \frac{8\pi}{3}
\end{aligned}
$$

Although it is tedious, it is possible to confirm the previous result using methods from Calculus II.

Example 5. Find the volume of the solid generated by rotation $r = 1 + \cos\theta,\ 0 \leq \theta \leq \pi$ about the x-axis.



It should be clear that this will generate a solid that has the same volume as the cardiod of revolution from Example 4. Now convert the polar equation to the equivalent rectangular equation.

$$r = 1 + \cos\theta$$

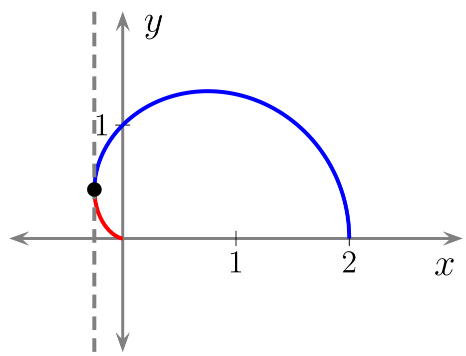
$$r^2 = r + r\cos\theta$$

$$x^2 + y^2 = \sqrt{x^2 + y^2} + x \tag{1}$$

Now if we "solve" for y as a function of x, for $y \geq 0$, we obtain 2 functions

$$f(x) = \sqrt{-x^2 + x + \frac{1}{2}\sqrt{4x+1} + \frac{1}{2}} \quad \text{and}$$

$$g(x) = \frac{\sqrt{-2x^2 + 2x - \sqrt{4x+1} + 1}}{\sqrt{2}}$$



We sketch the graph of $y = f(x)$ in blue and the graph of $y = g(x)$ in red. It is not difficult, with the help of the Implicit Function Theorem and (1), to determine that the vertical tangent line (shown as a dashed line in the above sketch) occurs at the point $(-1/4, \sqrt{3}/4)$.

It follows that the volume of the cardiod of revolution is given by

$$V = \underbrace{\pi \int_{-1/4}^{0} \left((f(x))^2 - (g(x))^2 \right) \, dx}_{V_1} + \underbrace{\pi \int_{0}^{2} (f(x))^2 \, dx}_{V_2}$$

Now

$$\begin{aligned} V_2 &= \pi \int_0^2 (f(x))^2 \, dx \\ &= \pi \int_0^2 \left(\sqrt{-x^2 + x + \frac{1}{2}\sqrt{4x+1} + \frac{1}{2}} \right)^2 \, dx \\ &= \pi \int_0^2 \left(-x^2 + x + \frac{1}{2}\sqrt{4x+1} + \frac{1}{2} \right) dx \\ &= \pi \left(\frac{-x^3}{3} + \frac{x^2}{2} + \frac{(4x+1)^{3/2}}{12} + \frac{x}{2} \right) \Big|_0^2 \\ &= \pi \left[\left(\frac{-2^3}{3} + \frac{2^2}{2} + \frac{(4(2)+1)^{3/2}}{12} + \frac{2}{2} \right) - \frac{1}{12} \right] \\ &= \frac{5\pi}{2} \end{aligned}$$

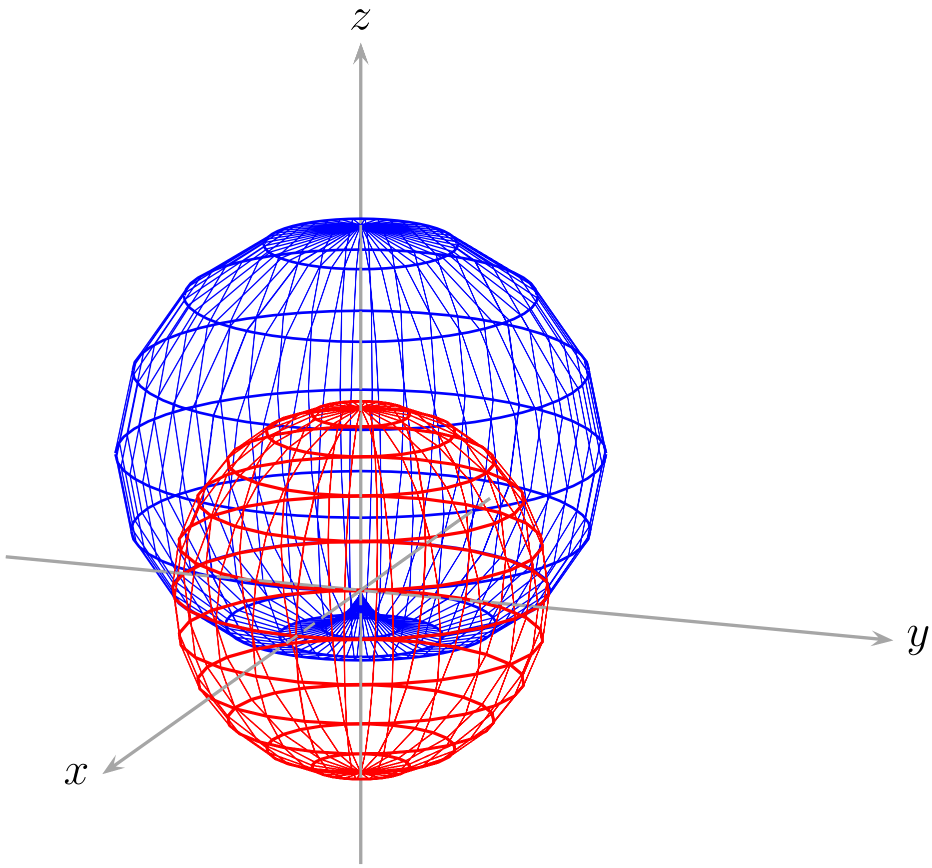
It turns out that $V_1 = \pi/6$ (we omit the calculation). Thus

$$V = V_1 + V_2 = \frac{5\pi}{2} + \frac{\pi}{6} = \frac{8\pi}{3}$$

as we saw above.

Example 6.

Let D be the solid bounded below by the hemisphere $\rho = 1,\ z \geq 0$, and above by the cardioid of revolution $\rho = 1 + \cos\phi$. Find the volume of D.



To summarize:

Cylindrical Coordinates: If $f(r, \theta, z)$ is continuous over a region $D \in \mathbb{R}^3$ then

$$dV = dz\, r\, dr\, d\theta$$

and

$$\begin{aligned} \iiint_D f\, dV &= \iiint_D f(r, \theta, z)\, dz\, r\, dr\, d\theta \\ &= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=h_1(\theta)}^{r=h_2(\theta)} \int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\, dz\, r\, dr\, d\theta \end{aligned}$$

Spherical Coordinates: If $f(\rho, \phi, \theta)$ is continuous over a region $D \in \mathbb{R}^3$ then

$$dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

and

$$\begin{aligned} \iiint_D f \, dV &= \iiint_D f(\rho, \phi, \theta) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \\ &= \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{min}}^{\phi=\phi_{max}} \int_{\rho=g_1(\phi,\theta)}^{\rho=g_2(\phi,\theta)} f(\rho, \phi, \theta) \, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta \end{aligned}$$

Coordinate Conversion Formulas

Cylindrical to Rectangular	Spherical to Rectangular	Spherical to Cylindrical
$x = r\ \cos\theta$	$x = \rho\ \sin\phi\ \cos\theta$	$r = \rho\ \sin\phi$
$y = r\ \sin\theta$	$y = \rho\ \sin\phi\ \sin\theta$	$z = \rho\ \cos\phi$
$z = z$	$z = \rho\ \cos\phi$	$\theta = \theta$