15.10 Change of Variables

Recall the formula for *change of variables* (u-substitution) from first semester calculus.

(1)
$$\int_a^b f(x)\,dx = \int_c^d f(g(u))g'(u)\,du$$

where $x = g(u)$, $a = g(c)$, and $b = g(d)$. We illustrate the method below.

Example 1. Evaluate the following integral.

(2)
$$\int_1^4 \frac{x}{\sqrt{1+3x^2}}\,dx$$

Although many students can evaluate the above integral directly, it is often advantageous to use the method called u-substitution to avoid possible arithmetic mistakes. A common choice would be to let $u = 1 + 3x^2$. Then $du = 6x\,dx$, $u(1) = 4$, and $u(4) = 49$ so that the integral in (2) becomes

$$\begin{aligned}\int_1^4 \frac{x}{\sqrt{1+3x^2}}\,dx &= \frac{1}{6}\int_1^4 \frac{6x\,dx}{\sqrt{1+3x^2}} \\ &= \frac{1}{6}\int_4^{49} \frac{du}{\sqrt{u}} \\ &= \frac{1}{3}\sqrt{u}\,\Big|_4^{49} = \frac{5}{3}\end{aligned}$$

Question - In the above example, it is clear that $f(x) = \frac{x}{\sqrt{1+3x^2}}$. What is $g(u)$?

We claim that $x = g(u) = +\sqrt{(u-1)/3}$. To see this, first note that $g(4) = 1$ and $g(49) = 4$. Also,

$$\frac{dx}{du} = g'(u) = \frac{1}{6}\,\frac{1}{\sqrt{(u-1)/3}}$$

so that

$$\begin{aligned} f(g(u))g'(u)\,du &= \frac{\sqrt{(u-1)/3}}{\sqrt{1+3(g(u))^2}}\,\frac{1}{6}\,\frac{1}{\sqrt{(u-1)/3}}\,du \\ &= \frac{1}{6}\,\frac{du}{\sqrt{u}} \end{aligned}$$

as expected.

We seek to find an analog to (1) for double (and eventually triple) integrals. In what follows, it will be helpful to make a few more observations about u-substitution.

1. The success of the method depends upon finding a suitable transformation, call it T, from an unknown interval $[c, d]$ to the interval $[a, b]$. In fact, we need T to be one-to-one with a continuous (nonzero) derivative.
2. The factor $\dfrac{dx}{du} = g'(u)$ seems to be very important.

So we need to find two-dimensional analogs of both T and the $\dfrac{dx}{du}$.

Consider the following example.

Example 2. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u,v) = (3u+2v, v) = (x,y)$.

We like to think of T as a map (function) from the uv-plane to the xy-plane. Now let $S = [0,1] \times [0,1]$. What is the image of S under the map T? In other words, what is $R = T(S)$?

We claim that R is the parallelogram with corners $(0,0), (3,0), (2,1), (5,1)$. In other words,

$$R = \{(x,y) : 2y \leq x \leq 2y+3,\ 0 \leq y \leq 1\}$$

Of course,

$$S = \{(u,v) : 0 \leq u \leq 1,\ 0 \leq v \leq 1\}$$

The mapping T in Example 2 is an example of a C^1-**transformation**. More specifically, we have the following.

Definition. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $T(u,v) = (g(u,v), h(u,v)) = (x,y)$. Then T is called a C^1-transformation if g and h have continuous first-order partial derivatives.

Now let's see how such a change of variables will affect a double integral. We continue with the previous example.

Example 3. Let $S, T,$ and $R = T(S)$ be as given in Example 2. Compare the integrals below if $f(x, y) = 1$.

$$\begin{aligned}\iint_R f(x,y)\,dx\,dy &= \iint_R 1\,dx\,dy \\ &= \text{area of } R \\ &= 3\end{aligned}$$

$$\begin{aligned}\iint_S f(x(u,v), x(u,v))\,du\,dv &= \iint_S 1\,du\,dv \\ &= \text{area of } S \\ &= 1\end{aligned}$$

It should come is no surprise that the integrals are not equal. We are missing whatever the analog of $\dfrac{dx}{du}$ in the two-dimensional case.

Definition. Let $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1-transformation defined by $T(u,v) = (g(u,v), h(u,v)) = (x,y)$. Then the Jacobian of T is defined by

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Example 4. Find the Jacobian of our running example.

$$x = 3u + 2v \quad \Longrightarrow \quad \frac{\partial x}{\partial u} = 3,\ \frac{\partial x}{\partial v} = 2$$

and

$$y = v \quad \Longrightarrow \quad \frac{\partial y}{\partial u} = 0,\ \frac{\partial y}{\partial v} = 1$$

It follows that $\dfrac{\partial(x,y)}{\partial(u,v)} = 3 - 0 = 3$. How convenient.

The next theorem is typically proven in an advanced calculus class.

Theorem 1. Suppose that T is a C^1-transformation whose Jacobian is nonzero and maps a region S in the uv-plane onto a region R in the xy-plane. Suppose that f is continuous on R and that S and R are "nice" regions. Finally, suppose that T is one-to-one, except perhaps on the boundary of S. Then

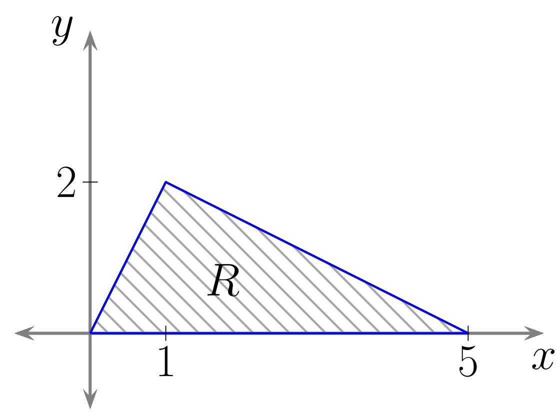
(3) $$\iint_R f(x,y)\,dA = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\,du\,dv$$

Example 5. Let R be the region bounded by the triangle with vertices $(0,0),(1,2),$ and $(5,0)$. Evaluate the integral below.

(4) $$\iint_R \cos\left(\frac{2x-y}{x+2y}\right)dx\,dy$$

Observe that the integral in (4) can be rewritten as

(5) $$\int_0^1\int_0^{2x} + \int_1^5\int_0^{\frac{5-x}{2}} \cos\left(\frac{2x-y}{x+2y}\right)dy\,dx$$

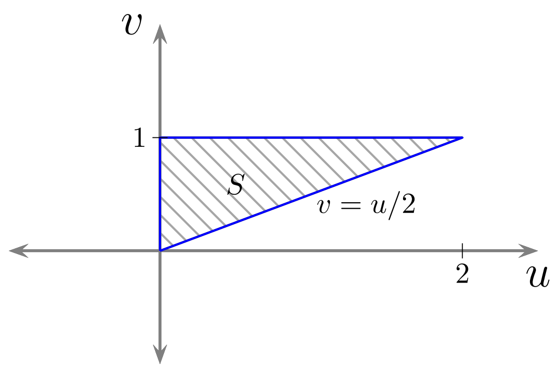


Unfortunately, the above integrals cannot be evaluated using elementary antiderivatives. Instead we try a change of variables. Let $u = \frac{2x-y}{c}$ and $v = \frac{x+2y}{c}$, for some constant c. The obvious choice is to set $c = 1$ but we can avoid a bunch of annoying fractions if we set $c = 5$.

Now let $T^{-1}(x,y) = \left(\frac{2x-y}{5}, \frac{x+2y}{5}\right)$ and let $S = T^{-1}(R)$.

Step 1. Find S.

Notice that $T^{-1}(0,0) = (0,0)$, $T^{-1}(1,2) = (0,1)$, and $T^{-1}(5,0) = (2,1)$. Since T^{-1} is linear in both coordinates, S will be a triangle in the uv-plane.



Step 2. Now find T. So consider the system

$$5u = 2x - y$$
$$5v = x + 2y$$

Multiplying the first equation by 2 and adding the resulting equation to the second yields

$$5x = 10u + 5v \quad \text{or} \quad x = 2u + v$$

Using a similar technique produces

$$y = 2v - u$$

In other words,

$$T(u,v) = (2u + v, 2v - u)$$

Step 3. Find the Jacobian.

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$
$$= (2)(2) - (-1)(1) = 5$$

Step 4. Now rewrite (4) using Theorem 1 and evaluate.

$$\iint_R \cos\left(\frac{2x-y}{x+2y}\right) dx\,dy = \iint_S \cos\left(\frac{u}{v}\right) 5\,du\,dv$$
$$= 5\int_0^1 \int_0^{2v} \cos\left(\frac{u}{v}\right) du\,dv$$
$$= 5\int_0^1 v\sin\left(\frac{u}{v}\right) \Big|_{u=0}^{u=2v} dv$$
$$= 5\sin 2 \int_0^1 v\,dv$$
$$= \frac{5\sin 2}{2} \approx 2.2732435671$$

Remark. Compare the result above with (5).

$$\int_0^1\int_0^{2x} \cos\left(\frac{2x-y}{x+2y}\right) dy\,dx \quad + \quad \int_1^5\int_0^{\frac{5-x}{2}} \cos\left(\frac{2x-y}{x+2y}\right) dy\,dx$$

Hint: Click on the above links and add the results.

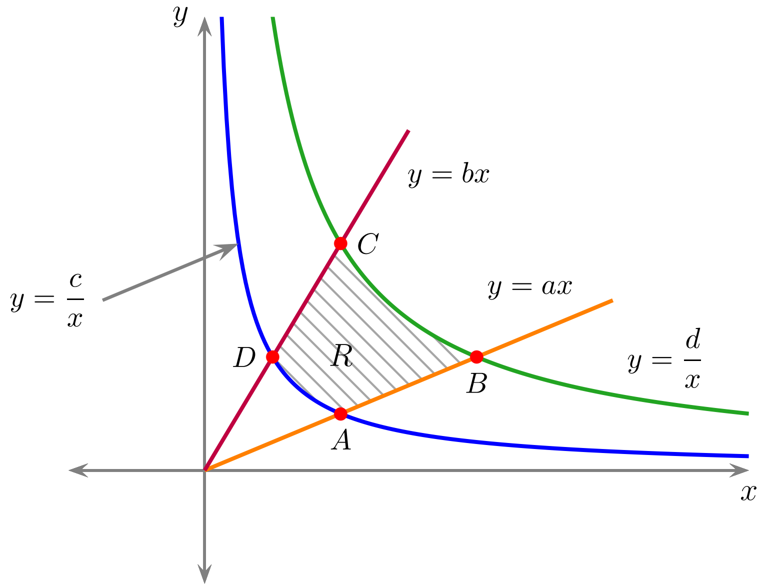


Figure 1: Region R bounded by two hyperbolas

Example 6. Let $b > a > 0$, $d > c > 0$, and let R be the shaded region shown in Figure 1. We leave it as an exercise to show that

$$A = A(\sqrt{c/a}, \sqrt{ac}), \;\; B = B(\sqrt{d/a}, \sqrt{ad})$$

$$C = C(\sqrt{d/b}, \sqrt{bd}), \;\; D = D(\sqrt{c/b}, \sqrt{bc})$$

Now let $T(u, v) = (u/v, v) = (x, y)$. Find T^{-1} and sketch the region $S = T^{-1}(R)$ in the uv-plane.

We claim that S is shaded region shown in Figure 2.

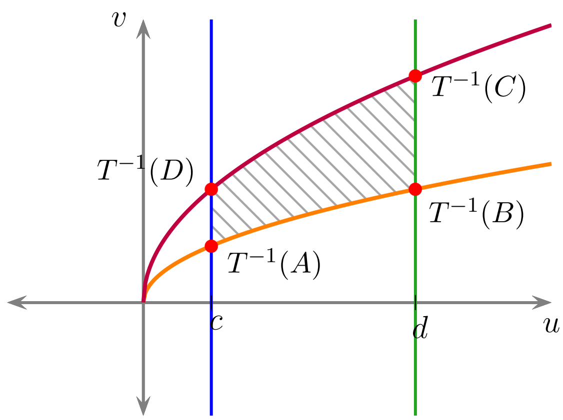


Figure 2: $S = T^{-1}(R)$

It is pretty easy to see that $T^{-1}(x, y) = (xy, y) = (u, v)$ so that, for example, $T^{-1}(A) = T^{-1}(\sqrt{c/a}, \sqrt{ac}) = (c, \sqrt{ac})$, etc. To see why radial lines are mapped to radical curves, let $x \geq 0$ and let $P = P(x, y)$ be an arbitrary point on the line $y = ax$. Then $P = (x, ax)$ and

$$\begin{aligned} T^{-1}(P) &= T^{-1}(x, ax) \\ &= (ax^2, ax) = (u, v) \end{aligned}$$

Rearranging the first coordinate equation yields

$$x = \sqrt{u/a}$$

It follows that

$$v = ax = a\sqrt{u/a} = \sqrt{au}$$

In other words, all of the points the orange radial line in Figure 1 get mapped to the radical function $v = \sqrt{au}$ as shown in Figure 2. Similarly, the points on the radial line $y = bx$ get mapped to the radical function $v = \sqrt{bu}$.

Now let $Q = (x, d/x),\ x > 0$ be an arbitrary point on the hyperbola $y = d/x$ (shown in green in Fig. 1). Then

$$\begin{aligned} T^{-1}(Q) &= T^{-1}(x, d/x) \\ &= (d, d/x) \end{aligned}$$

In other words, all of the quadrant I points on the hyperbola $y = d/x$ get mapped to the vertical line $u = d$. See the green line in Figure 2.

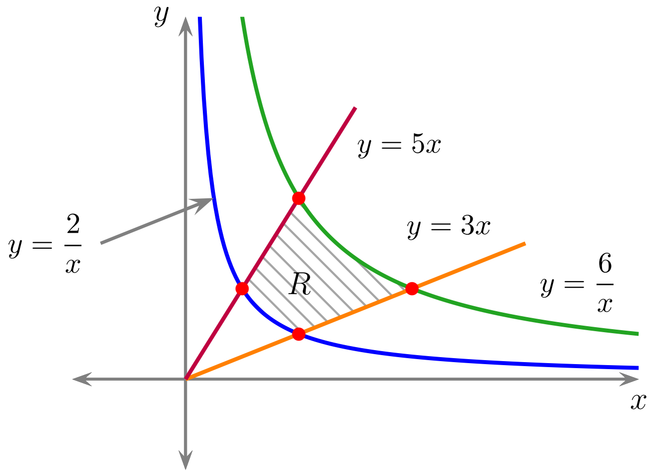


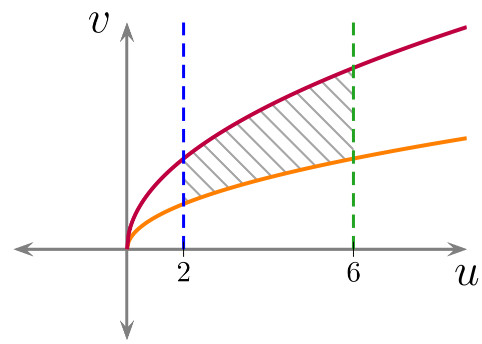
Figure 3: Region R from Example 7 (not to scale)

Example 7. Find the area of the shaded region R shown in Figure 3.

Continuing with the notation from the previous example, we have

$$\text{Area of } R = \iint_R 1\, dA = \iint_S 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv$$

We leave it as an exercise to show that $\frac{\partial(x,y)}{\partial(u,v)} = 1/v$. It follows from the previous example, that



$$\iint_S 1 \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv = \int_2^6 \int_{\sqrt{3u}}^{\sqrt{5u}} \frac{1}{v}\, dv\, du$$

$$= \int_2^6 \ln v \Big|_{v=\sqrt{3u}}^{v=\sqrt{5u}} du$$

$$= \frac{\ln 5/3}{2} \int_2^6 du$$

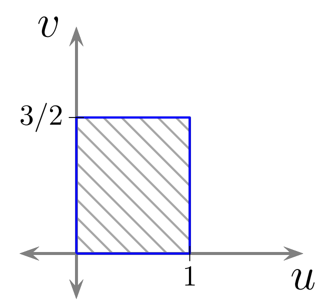
$$= 2 \ln 5/3$$

Example 8. Let R be the quadrilateral with vertices $(0,0),(1,-1)$, $(5/2,1/2),(3/2,3/2)$. Evaluate the following integral.

(6) $$\iint_R (x+y)e^{x^2-y^2}\,dx\,dy$$

This doesn't look too friendly. However, notice that since the exponent factors as $(x-y)(x+y)$, so we try $u=\frac{x-y}{2}$ and $v=\frac{x+y}{2}$. It is easy to show that this implies $x=u+v$ and $y=v-u$. It is routine to show that $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|=2$ and that if we let $T^{-1}(x,y)=\left(\frac{x-y}{2},\frac{x+y}{2}\right)$, then $S=T^{-1}(R)$ is a rectangle in the uv-plane with vertices $(0,0),(1,0)$, $(1,3/2),(0,3/2)$. See the figure below.

Thus



$$\begin{aligned}\iint_R (x+y)e^{x^2-y^2}\,dx\,dy &= \iint_S 2ve^{4uv}\,2\,du\,dv\\ &= \int_0^{3/2}\int_0^1 4ve^{4uv}\,du\,dv\\ &= \int_0^{3/2} e^{4uv}\Big|_{u=0}^{u=1}\,dv\\ &= \int_0^{3/2}(e^{4v}-1)\,dv\\ &= \frac{e^6-7}{4}\end{aligned}$$

Remark. Notice that we must use integration by parts if we wish to integrate with respect to v first.