16.7 Surface Integrals

Let f be a function defined on a region of $\mathbb{R}^3$ that contains the surface S. In this section we will define the surface integral of f over S.

Definition. Suppose that the surface S has the vector equation

(1) $$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in D$$

Now if the components of $\mathbf{r}_u$ and $\mathbf{r}_v$ are continuous and $\mathbf{r}_u$ and $\mathbf{r}_v$ are nonzero and nonparallel in the interior of D, then we define the **surface integral of f over the surface S** by

(2) $$\iint_S f(x,y,z)\,dS = \iint_D f\left(\mathbf{r}(u,v)\right)\left|\mathbf{r}_u \times \mathbf{r}_v\right|\,dA$$

Remark. Compare with the definition of a line integral from section 16.2.

(3) $$\int_C f(x,y,z)\,ds = \int_a^b f\left(\mathbf{r}(t)\right)\left|\mathbf{r}'(t)\right|dt$$

Here $\mathbf{r}(t)$ is a smooth parametrization of the space curve C.

Example 1. Let $b > 0$ and let S be the cone with vector equation

$$\mathbf{r}(u,v) = u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}, \quad 0 \le u \le b,\ 0 \le v \le 2\pi$$

Evaluate the surface integral $\iint_S x^2 z\, dS$.

Now

$$\mathbf{r}_u = \cos v\,\mathbf{i} + \sin v\,\mathbf{j} + \mathbf{k}, \quad \mathbf{r}_v = -u\sin v\,\mathbf{i} + u\cos v\,\mathbf{j}$$

So that

$$\mathbf{r}_u \times \mathbf{r}_v = -u\cos v\,\mathbf{i} + u\sin v\,\mathbf{j} + u\,\mathbf{k}$$

and

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} \\ &= \sqrt{2}u \end{aligned}$$

since $u \ge 0$. Thus

(4)
$$\begin{aligned} \iint_S x^2 z\, dS &= \sqrt{2}\iint_D \left(u^2\cos^2 v\right) u \cdot u\, dA \\ &= \sqrt{2}\int_0^b u^4\, du \int_0^{2\pi} \cos^2 v\, dv \\ &= \frac{\sqrt{2}\,b^5}{5}\int_0^{\pi} \frac{1+\cos 2v}{2}\, dv \\ &= \frac{b^5}{5\sqrt{2}}\left(v + \frac{\sin 2v}{2}\right)\Bigg|_0^{2\pi} \\ &= \frac{\pi\sqrt{2}\,b^5}{5} \end{aligned}$$

Let's rework the previous example with the parametrization (of S) given by the vector equation

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{x^2+y^2}\,\mathbf{k}, \quad x^2+y^2 \le b^2$$

Then

$$\mathbf{r}_x = \mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\,\mathbf{k}, \quad \mathbf{r}_y = \mathbf{j} + \frac{y}{\sqrt{x^2+y^2}}\,\mathbf{k}$$

Thus

$$\mathbf{r}_x \times \mathbf{r}_y = \frac{-x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \frac{-y}{\sqrt{x^2+y^2}}\,\mathbf{j} + \mathbf{k}$$

and

$$|\mathbf{r}_x \times \mathbf{r}_y| = \sqrt{2}$$

Now

$$\iint_S x^2 z\,dS = \sqrt{2}\iint_D x^2\sqrt{x^2+y^2}\,dA$$

Switching to polar coordinates we obtain

$$= \sqrt{2}\int_0^{2\pi}\int_0^b \left(r^2\cos^2\theta\right) r\cdot r\,dr\,d\theta$$

which is (4).

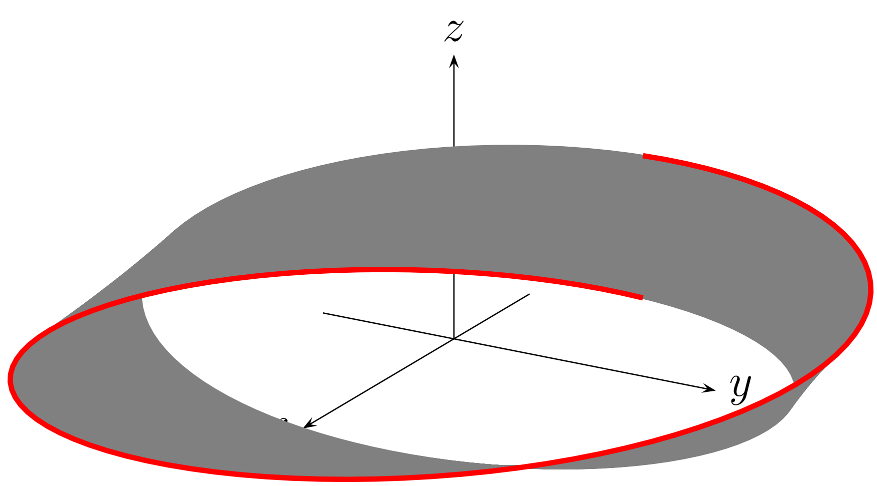
Orientation

We call a smooth surface S **orientable** or **two-sided** if it is possible to define a field $\mathbf{n}$ of unit normal vectors on S that varies continuously with position.

Smooth surfaces that enclose solids are orientable and by convention, $\mathbf{n}$ is chosen to point outward. At each point on an orientable surface the vector $\mathbf{n}$ indicates the positive direction.

Example 2. Nonorientable Surface

The **Mobius Strip** is an example if a nonorientable (or one-sided) surface.



Now let S be a two-sided surface with vector equation

$$\mathbf{r}(u,v) = x(u,v)\,\mathbf{i} + y(u,v)\,\mathbf{j} + z(u,v)\,\mathbf{k}, \quad (u,v) \in D$$

Now suppose that S has a tangent plane at every point (except possibly at a boundary point). Then it is not difficult to show that

$$\mathbf{n}_1 = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

and $\mathbf{n}_2 = -\mathbf{n}_1$ are unit vectors *normal* to the surface S.

If S encloses a solid region E, it is conventional that the **positive orientation** is one for which the unit normal vectors point away from E.

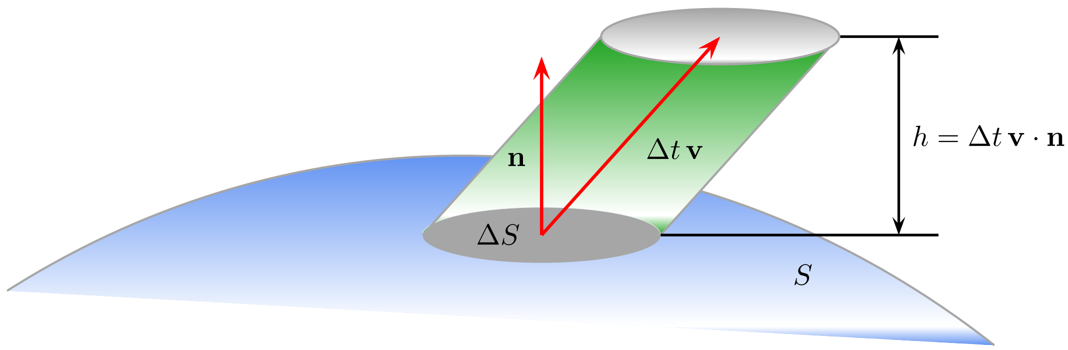


Figure 1: Fluid flow across surface S.

Surface Integrals for Vector Fields

Let D be a region in space bounded by a closed surface S. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid flowing smoothly through D and $\delta = \delta(t, x, y, z)$ be the density of the fluid at (x, y, z) at time t. Consider the vector field $\mathbf{F} = \delta\mathbf{v}$ and suppose all functions in question have continuous first partial derivatives.

Now consider a small patch ΔS on the surface S (See Figure 1). If Δt is small, then the volume ΔV of fluid that crosses the patch is approximately equal to the volume of a the cylinder with base area ΔS times the height $h = (\Delta t\mathbf{v}) \cdot \mathbf{n}$.

We have

$$\Delta V \approx \mathbf{v} \cdot \mathbf{n}\, \Delta S\, \Delta t$$

So the mass of this volume of fluid is

$$\Delta m \approx \delta\mathbf{v} \cdot \mathbf{n}\, \Delta S\, \Delta t = \mathbf{F} \cdot \mathbf{n}\, \Delta S\, \Delta t$$

It follows that the rate at which mass is leaving the region D across the patch ΔS is roughly

$$\frac{\Delta m}{\Delta t} \approx \mathbf{F} \cdot \mathbf{n}\, \Delta S$$

Summing over S yields

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \mathbf{F} \cdot \mathbf{n}\, \Delta S \tag{5}$$

So the right-hand side of this last equation gives an estimate of the average rate at which mass flows across S.

Notice that the right-hand side of (5) is a Riemann sum. So if S and $\mathbf{F}$ are nice enough, $\Delta t \to 0$ and $\Delta S \to 0$ produces

$$\frac{dm}{dt} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS$$

This leads to the following definition.

Definition. The Surface Integral of $\mathbf{F}$ over S

If $\mathbf{F}$ is a continuous vector filed defined on an oriented surface S with unit normal vector $\mathbf{n}$, then the **surface integral of $\mathbf{F}$ over S** is

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS \tag{6}$$

This integral is also called the **flux** of $\mathbf{F}$ across S.

If S is defined by a vector function $\mathbf{r}(u,v)$ over some domain D, then

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad \text{or} \quad \mathbf{n} = -\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$$

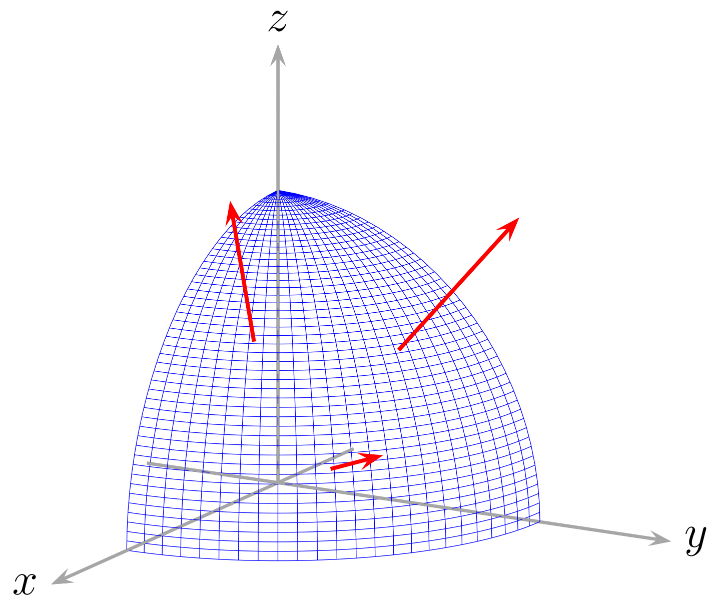
Here we choose the quantity that gives us the preferred direction.

It follows that

$$\begin{aligned}
\iint_S \mathbf{F} \cdot d\mathbf{S} &= \iint_S \mathbf{F} \cdot \mathbf{n} \, dS \\
&= \iint_S \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \, dS \\
&= \iint_D \left[\mathbf{F}\left(\mathbf{r}(u,v)\right) \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \right] |\mathbf{r}_u \times \mathbf{r}_v| \; dA \\
&= \iint_D \mathbf{F}\left(\mathbf{r}(u,v)\right) \cdot \left(\mathbf{r}_u \times \mathbf{r}_v\right) \, dA \qquad (7)
\end{aligned}$$

Example 3.

Find the flux of the field $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$ across the portion of the sphere $x^2 + y^2 + z^2 = a^2$ in the first octant in the direction away from the origin.



Let $g(x, y, z) = x^2 + y^2 + z^2$. Then the given surface, call it S, is just the level surface $g = a^2$. Observe that S can defined by the vector equation

$$\mathbf{r}(\phi, \theta) = a \sin\phi\, \cos\theta\, \mathbf{i} + a \sin\phi\, \sin\theta\, \mathbf{j} + a \cos\phi\, \mathbf{k},\ (\phi, \theta) \in D$$

and

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = a^2 \sin^2\phi\, \cos\theta\, \mathbf{i} + a^2 \sin^2\phi\, \sin\theta\, \mathbf{j} + a^2 \sin\phi\, \cos\phi\, \mathbf{k}$$

Here $D = \{(\phi, \theta) : 0 \le \phi \le \pi/2, 0 \le \theta \le \pi/2\}$. (See examples 4 and 10 from section 16.6 in the text.)

It follows that,

$$\begin{aligned}\iint_S \mathbf{F}\cdot d\mathbf{S} &= \iint_D \mathbf{F}\left(\mathbf{r}(u,v)\right)\cdot\left(\mathbf{r}_u\times\mathbf{r}_v\right)\,dA\\
&= \iint_D \left(a\sin\phi\,\sin\theta\,\mathbf{i} + a\sin\phi\,\cos\theta\,\mathbf{j}\right)\cdot\\
&\qquad \left(a^2\sin^2\phi\,\cos\theta\,\mathbf{i} + a^2\sin^2\phi\,\sin\theta\,\mathbf{j} + a^2\sin\phi\,\cos\phi\,\mathbf{k}\right)\,dA\\
&= a^3\int_0^{\pi/2}\int_0^{\pi/2}\left(\sin^3\phi\,\sin\theta\,\cos\theta + \sin^3\phi\,\sin\theta\,\cos\theta\right)\,d\phi\,d\theta\\
&= a^3\int_0^{\pi/2}\sin^3\phi\,d\phi\int_0^{\pi/2}2\sin\theta\,\cos\theta\,d\theta\\
&= \vdots\\
&= \frac{2a^3}{3}\end{aligned}$$

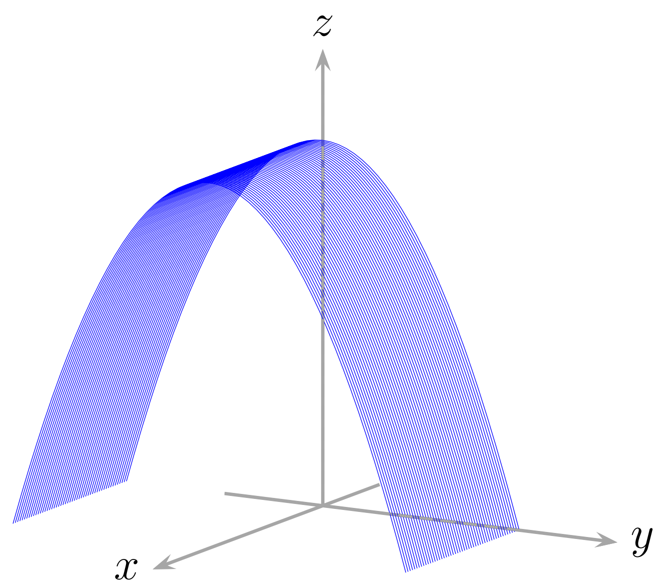
Redo the last example by observing that S can also be defined by the vector equation

$$\mathbf{r}(x,y) = x\,\mathbf{i} + y\,\mathbf{j} + \sqrt{a^2-x^2-y^2}\,\mathbf{k},\quad (x,y)\in D,$$

where $D = \{(x,y)\colon\ x^2+y^2\le a^2, x\ge 0, y\ge 0\}$.

Example 4.

Find the flux of the field $\mathbf{F} = z^2\,\mathbf{i} + x\,\mathbf{j} - 3z\,\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0,\ x = 1,$ and $z = 0$.



Let S be the given parabolic cylinder and $D = \{(x, y) : 0 \le x \le 1, -2 \le y \le 2\}$. Then S can be defined by the vector equation

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (4 - y^2)\,\mathbf{k}, \quad (x, y) \in D$$

Proceeding as usual we have

$$\mathbf{r}_x = \mathbf{i} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} - 2y\,\mathbf{k}$$

So that

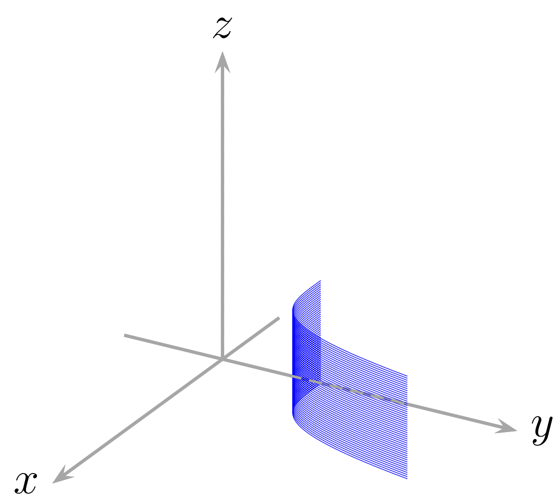
$$\mathbf{r}_x \times \mathbf{r}_y = 2y\,\mathbf{j} + \ \mathbf{k}$$

Following (7) we obtain

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_D \mathbf{F}\left(\mathbf{r}(x,y)\right) \cdot \left(\mathbf{r}_x \times \mathbf{r}_y\right)\, dA \\
&= \iint_D \left(z^2\,\mathbf{i} + x\,\mathbf{j} - 3(4-y^2)\,\mathbf{k}\right) \cdot \left(2y\,\mathbf{j} + \mathbf{k}\right)\, dA \\
&= \int_{-2}^{2}\int_0^1 (3y^2 + 2xy - 12)\, dx\, dy \\
&= \vdots \\
&= -32
\end{aligned}$$

Example 5.

Let S be the portion of the cylinder $y = e^x$ in the first octant, with $0 \leq x \leq 1$ and $0 \leq z \leq 1$. And let $\mathbf{n}$ be the unit vector normal to S that points away from the yz-plane. Find the flux of the field $\mathbf{F} = -2\,\mathbf{i} + 2y\,\mathbf{j} + z\,\mathbf{k}$ across S in the direction of $\mathbf{n}$.



Let $D = \{(x, z) : 0 \leq x \leq 1, 0 \leq z \leq 1\}$. Then S can be defined by the vector equation

$$\mathbf{r}(x, z) = x\,\mathbf{i} + e^x\,\mathbf{j} + z\,\mathbf{k}, \quad (x, z) \in D$$

Proceeding as before we have

$$\mathbf{r}_x = \mathbf{i} + e^x\,\mathbf{j} \quad \text{and} \quad \mathbf{r}_z = \mathbf{k}$$

So that

$$\mathbf{r}_x \times \mathbf{r}_z = e^x\,\mathbf{i} - \mathbf{j}$$

From (7) we have

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_D \mathbf{F}\left(\mathbf{r}(x,z)\right)\cdot\left(\mathbf{r}_x\times\mathbf{r}_z\right)\,dA\\ &= \iint_D \left(-2\,\mathbf{i}+2e^x\,\mathbf{j}+z\,\mathbf{k}\right)\cdot\left(e^x\,\mathbf{i}-\,\mathbf{j}\right)\,dA\\ &= -4\int_0^1\int_0^1 e^x\,dx\,dz\\ &= -4\int_0^1 e^x\,dx\\ &= 4(1-e)\end{aligned}$$

The given surface can also be parameterized by the vector equation

(8) $$\mathbf{r}_1(y,z)=\ln y\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k},\quad (y,z)\in D_1$$

Exercise - Set up and evaluate an equivalent integral for Example 5 using the vector equation (8). Of course, you will need to find D_1.

Example 6. Let $b > 0$. Find the outward flux of the field

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the surface of the sphere $S\colon x^2 + y^2 + z^2 = \rho^2 = b^2$.

Notice that the outward normal is

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2 + y^2 + z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{b}$$

and

$$\mathbf{F} \cdot \mathbf{n} = \frac{x^2 + y^2 + z^2}{b^4} = \frac{1}{b^2}$$

It follows that the flux across the outer surface S is

$$\begin{aligned}\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_S \frac{1}{b^2}\, dS = \frac{1}{b^2} \iint_S dS \\ &= \frac{1}{b^2} \times \text{surface area of } S \\ &= \frac{1}{b^2}\, 4\pi b^2 = 4\pi\end{aligned}$$

So the outward flux across a sphere of *any* radius for this vector field is 4π. We will have more to say about this example in section 16.9.