16.9 The Divergence Theorem

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$. In the previous section we saw that the circulation density for $\mathbf{F}$ was given by

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$

We went on to generalize the *circulation-curl* (tangential) form of Green's Theorem and obtained something called Stokes' Theorem.

In this section we'll try the same thing with the *flux-divergence* (normal) form of the Green's Theorem.

Divergence in Three Dimensions

Let $\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} + P\,\mathbf{k}$. Recall that the flux density (divergence) is given by

$$\begin{aligned} \operatorname{div}\mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \end{aligned}$$

Theorem 1. The Divergence Theorem

Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. Then the flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ is given by

(1) $$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

where D is the region enclosed by the surface S.

Notice that the integrand of surface integral (LHS) is the component of the vector field $\mathbf{F}$ in the direction normal to the surface S while the integrand of the volume integral (RHS) is the *flux-density* ($\operatorname{div}\mathbf{F}$) over the region D. (Compare this to the **Normal Form** of Green's Theorem.)

Remark. When the region D is the focus of attention, we often refer to S as its boundary. In such cases we will use the notation ∂D to refer to the "boundary of D".

Example 1. Using the Divergence Theorem

Let $\mathbf{F} = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$. Find the outward flux across the boundary of D if D is the cube in the first octant bounded by $x = 1,\ y = 1,\ z = 1$.

According to the Divergence Theorem

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

The RHS calculation is very straight forward.

$$\begin{aligned}
\iiint_D \nabla\cdot\mathbf{F}\,dV &= \int_0^1\int_0^1\int_0^1 (2x+2y+2z)\,dx\,dy\,dz \\
&= \int_0^1\int_0^1 (1+2y+2z)\,dy\,dz \\
&= \int_0^1 (2+2z)\,dz \\
&= 3
\end{aligned}$$

So the outward flux is 3.

Let's calculate the outward flux directly by evaluating the LHS of (1). To do this we need to evaluate the surface integrals on the six faces of the cube. Let S_1 be the face that lies in the plane $z = 1$.

Then S_1 is given by the vector equation

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + \mathbf{k}, \quad (x, y) \in R = [0, 1] \times [0, 1]$$

and

$$\mathbf{r}_x \times \mathbf{r}_y = \mathbf{i} \times \mathbf{j} = \mathbf{k} = \mathbf{n}$$

It follows that

$$\begin{aligned}\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS &= \iint_R \left(x^2\,\mathbf{i} + y^2\,\mathbf{j} + 1^2\,\mathbf{k}\right) \cdot \mathbf{k}\, dA \\ &= \iint_R dA \\ &= \text{area of } R \\ &= 1\end{aligned}$$

It is easy to show that the corresponding surface integrals on the faces that intersect $x = 1$ and $y = 1$ are the same. Finally,

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 1 + 1 + 1 + 0 + 0 + 0 = 3$$

since the surface integrals across the faces of the cube that lie in any of the coordinate planes is zero.

Example 2. Flux Across a Thick Sphere

Let

$$\mathbf{F} = (5x^3 + 12xy^2)\,\mathbf{i} + (y^3 + e^y \sin z)\,\mathbf{j} + (5z^3 + e^y \cos z)\,\mathbf{k}$$

Find the outward flux across the boundary of the region D where D is "the solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$".

We appeal to the Divergence Theorem.

$$\frac{\partial(5x^3 + 12xy^2)}{\partial x} = 15x^2 + 12y^2$$

$$\frac{\partial(y^3 + e^y \sin z)}{\partial y} = 3y^2 + e^y \sin z$$

$$\frac{\partial(5z^3 + e^y \cos z)}{\partial z} = 15z^2 - e^y \sin z$$

Hence

$$\begin{aligned}\operatorname{div} \mathbf{F} &= \nabla \cdot \mathbf{F} \\ &= 15x^2 + 12y^2 + 3y^2 + e^y \sin z + 15z^2 - e^y \sin z \\ &= 15x^2 + 15y^2 + 15z^2\end{aligned}$$

It follows that

$$
\begin{aligned}
\text{flux} &= \iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS \\
&= \iiint_D \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_D \left(15x^2 + 15y^2 + 15z^2\right) dx\, dy\, dz \\
&= 15 \int_0^{2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=1}^{\rho=\sqrt{2}} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 15(2\pi) \int_{\phi=0}^{\phi=\pi} \sin\phi \int_{\rho=1}^{\rho=\sqrt{2}} \rho^4\, d\rho\, d\phi \\
&= 15 \int_0^{2\pi} \int_{\phi=0}^{\phi=\pi} \int_{\rho=1}^{\rho=\sqrt{2}} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= 15 \left(\int_0^{2\pi} d\theta\right) \left(\int_{\phi=0}^{\phi=\pi} \sin\phi\, d\phi\right) \left(\int_{\rho=1}^{\rho=\sqrt{2}} \rho^4\, d\rho\right) \\
&\vdots \\
&= 15 \times 2\pi \times 2 \times \frac{4\sqrt{2}-1}{5}
\end{aligned}
$$

Example 3.

Let D be the (elliptic) cylindrical solid bounded by $4x^2 + y^2 = 4$ and the planes $z = 0$ and $z = 2$. Evaluate the following integral.

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS$$

where

$$\mathbf{F} = x^3 \, \mathbf{i} + y^3 \, \mathbf{j} + z^2 \, \mathbf{k}$$

Once again we will use the Divergence Theorem. Notice that

$$\nabla \cdot \mathbf{F} = 3x^2 + 3y^2 + 2z$$

so that

$$\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = \iiint_D \left(3x^2 + 3y^2 + 2z\right) dV$$

Can you rewrite this as an iterated integral? *Hint: Although cylindrical coordinates look tempting, it turns out to be easier to integrate using rectangular coordinates.*

Thus

$$
\begin{aligned}
\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS &= \iiint_D \left(3x^2 + 3y^2 + 2z\right) dV \\
&= \int_{-1}^{1} \int_{-2\sqrt{1-x^2}}^{2\sqrt{1-x^2}} \int_0^2 (3x^2 + 3y^2 + 2z) \, dz \, dy \, dx \\
&= 4 \int_0^1 \int_0^{2\sqrt{1-x^2}} \int_0^2 (3x^2 + 3y^2 + 2z) \, dz \, dy \, dx \\
&= 4 \int_0^1 \int_0^{2\sqrt{1-x^2}} (6x^2 + 6y^2 + 4) \, dy \, dx \\
&= 4 \int_0^1 (6x^2 y + 2y^3 + 4y) \Big|_{y=0}^{y=2\sqrt{1-x^2}} \, dx \\
&= 16 \int_0^1 \left(3x^2\sqrt{1-x^2} + 4(1-x^2)^{3/2} + \sqrt{1-x^2}\right) \, dx
\end{aligned}
$$

which can be evaluated using a routine trig substitution to yield

$$
\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = 3\pi + 12\pi + 8\pi = 23\pi
$$

Remark. For the curious, integrating in cylindrical coordinates results in

$$
\iint_{\partial D} \mathbf{F} \cdot \mathbf{n} \, dS = 96 \int_0^{2\pi} \frac{1}{1 + 3\cos^2\theta} \, d\theta + 32 \int_0^{2\pi} \frac{1}{(1 + 3\cos^2\theta)^2} \, d\theta
$$

and the last integral is non-trivial.

Example 4.

Find the net outward flux of the field

$$\mathbf{F} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\rho^3}, \qquad \rho = \sqrt{x^2 + y^2 + z^2}$$

across the boundary of the region $D\colon 0 < a^2 \le \rho^2 \le b^2$. *Note:* This is another “thick sphere” computation, as we saw in Example 2.

Now by the Divergence Theorem, we can calculate the flux by evaluating the integral $\iiint_D \nabla \cdot \mathbf{F}\,dV$. Observe that

$$\frac{\partial \rho}{\partial x} = \frac{x}{\sqrt{x^2 + y^2 + z^2}} = \frac{x}{\rho}$$

It follows that

$$\begin{aligned} \frac{\partial (x/\rho^3)}{\partial x} &= \frac{\rho^3 - 3x\rho^2(x/\rho)}{\rho^6} \\ &= \frac{\rho}{\rho}\,\frac{\rho^2 - 3x^2}{\rho^5} = \frac{-2x^2 + y^2 + z^2}{\rho^5} \end{aligned}$$

Similarly,

$$\frac{\partial (y/\rho^3)}{\partial y} = \frac{-2y^2 + x^2 + z^2}{\rho^5} \quad \text{and} \quad \frac{\partial (z/\rho^3)}{\partial z} = \frac{-2z^2 + x^2 + y^2}{\rho^5}$$

It follows that

$$\nabla \cdot \mathbf{F} = 0 \tag{2}$$

Hence

$$\iiint_D \nabla \cdot \mathbf{F}\,dV = 0$$

So the flux leaving the region via the inner surface is the negative of the flux leaving the region through the outer surface. It seems worthwhile to compute the flux across one the surfaces directly (e.g., see Example 16.7.6). Notice that the outward normal is

$$\mathbf{n} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{b}$$

and

$$\mathbf{F}\cdot\mathbf{n} = \frac{x^2+y^2+z^2}{b^4} = \frac{1}{b^2}$$

It follows that the flux across the outer surface S_b is

$$\begin{aligned}\iint_{S_b} \mathbf{F}\cdot\mathbf{n}\,dS &= \iint_{S_b} \frac{1}{b^2}\,dS = \frac{1}{b^2}\iint_{S_b} dS \\ &= \frac{1}{b^2} \times \text{surface area of } S_b \\ &= \frac{1}{b^2}\,4\pi b^2 = 4\pi\end{aligned}$$

So the outward flux across any sphere centered at the origin is 4π. In fact, the above result is not limited spheres but applies to any smooth bounded region that includes the origin.

We explore this result below.

Remark. According to (2) and the Divergence Theorem, the flux across S_b should be 0. Can you explain the nonzero result above?

Gauss's Law

Suppose that a point charge* q (measured in *coulombs*) is placed at the origin. In electromagnetic theory, such a point charge would create an electric field emanating from the origin whose vector equation is given by

(3) $$\mathbf{E}(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{q}{\mathbf{r}^2}\frac{\mathbf{r}}{|\mathbf{r}|} = \frac{q}{4\pi\epsilon_0}\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{\rho^3}$$

in units of *Newtons per coulomb* (or volts per meter). Here ϵ_0 is a physical constant (called the permittivity of free space) and $\mathbf{r}$ is the position vector of a point in space. Using the notation from the last example, we can rewrite (3) as

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\mathbf{F}$$

It follows by Example 4 that the flux across any reasonable surface S that includes the origin is q/ϵ_0. To see this we consider a sphere S_b of radius b centered at the origin that contains S. Then

$$\iint_{S_b}\mathbf{E}\cdot\mathbf{n}\,dS = \frac{q}{4\pi\epsilon_0}\iint_{S_b}\mathbf{F}\cdot\mathbf{n}\,dS = \frac{q}{4\pi\epsilon_0}\times 4\pi = \frac{q}{\epsilon_0}$$

Notice that

(4) $$\nabla\cdot\mathbf{E} = \nabla\cdot\frac{q}{4\pi\epsilon_0}\mathbf{F} = \frac{q}{4\pi\epsilon_0}\nabla\cdot\mathbf{F} = 0$$

* - For example, the charge from a single proton is exactly $1.602176634\times10^{-19}$ coulombs.

Now let D be the region between S and S_b. If the Divergence Theorem applies over the region D, then, because of (4), the net outward flux of $\mathbf{E}$ across the boundary of D must be zero. But the outward flux across S_b is q/ϵ_0. It follows that the outward flux across S must be the same.

This is Gauss's Law:

(5) $$\iint_S \mathbf{E} \cdot \mathbf{n} \, dS = \frac{q}{\epsilon_0}$$

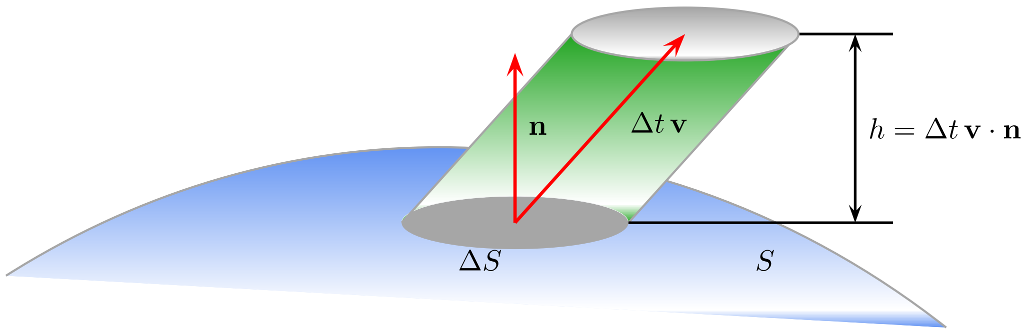

Figure 1: Fluid flow across surface S.

Continuity Equation of Hydrodynamics

Let D be a region in space bounded by a closed surface S. Let $\mathbf{v}(x, y, z)$ be the velocity field of a fluid flowing smoothly through D and $\delta = \delta(t, x, y, z)$ be the density of the fluid at (x, y, z) at time t. Consider the vector field $\mathbf{F} = \delta\mathbf{v}$ and suppose all functions in question have continuous first partial derivatives.

We first observe that the surface integral $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ is the rate at which mass leaves the region D across S. To see this, consider a small patch ΔS on the surface S (See Figure 1). If Δt is small, the volume ΔV of fluid that crosses the patch is approximately equal to the volume of a cylinder with base area ΔS times the height $h = (\Delta t\mathbf{v})\cdot\mathbf{n}$.

Notice that

$$\Delta V \approx \mathbf{v}\cdot\mathbf{n}\,\Delta S\,\Delta t$$

And the mass of this volume of fluid is

$$\Delta m \approx \delta\mathbf{v}\cdot\mathbf{n}\,\Delta S\,\Delta t$$

So the rate at which mass is leaving the region D across the patch ΔS is roughly

$$\frac{\Delta m}{\Delta t} \approx \delta \mathbf{v} \cdot \mathbf{n} \, \Delta S$$

It follows that

$$\frac{\sum \Delta m}{\Delta t} \approx \sum \delta \mathbf{v} \cdot \mathbf{n} \, \Delta S$$

is an estimate of the average rate at which mass flows across S. Letting $\Delta t \to 0$ and $\Delta S \to 0$ produces

$$\frac{dm}{dt} = \iint_S \delta \mathbf{v} \cdot \mathbf{n} \, dS = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS$$

Now let B be a ball (solid sphere) in D centered at some point Q. Then the average value of $\nabla \cdot \mathbf{F}$ is given by

$$\frac{1}{\text{Vol}(B)} \iiint_B \nabla \cdot \mathbf{F} \, dV$$

By the assumptions on the vector field, $\nabla \cdot \mathbf{F}$ is continuous. Hence, there is a point $P \in B$ such that

$$(\nabla \cdot \mathbf{F})_P = \frac{1}{\text{Vol}(B)} \iiint_B \nabla \cdot \mathbf{F} \, dV = \frac{1}{\text{Vol}(B)} \iint_{\partial B} \mathbf{F} \cdot \mathbf{n} \, dS$$

$$= \frac{\text{rate at which mass leaves } B \text{ across its boundary } \partial B}{\text{volume of } B}$$

The last expression gives the decrease in mass per unit volume.

Holding Q fixed and letting the radius of the ball go to zero yields

$$(\nabla \cdot \mathbf{F})_Q = -\left(\frac{\partial \delta}{\partial t}\right)_Q$$

And since Q was arbitrary, we have

$$\nabla \cdot \mathbf{F} = -\frac{\partial \delta}{\partial t}$$

The last equation, called the continuity equation of hydrodynamics, is often written as

$$\nabla \cdot \mathbf{F} + \frac{\partial \delta}{\partial t} = 0 \tag{6}$$

It says that the divergence of $\mathbf{F}$ at a point Q is the rate at which the density of the fluid is decreasing at Q. So the Divergence Theorem

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_D \nabla \cdot \mathbf{F}\, dV$$

says that the net decrease in fluid density in D is accounted for by mass transported across the boundary of D. In other words, the theorem is about conservation of mass.

Remark. There are numerous continuity equations similar to Equation (6) throughout physics.

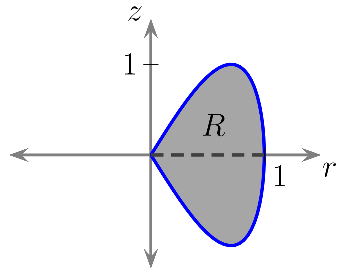

Figure 2: Region R

Example 5. Consider the region R bounded by the space curve defined below (see Fig. 2).

(7) $$z = \sin 2t \text{ and } r = \sin t, \quad 0 \leq t \leq \pi$$

Find the volume of the solid E generated by rotating the region R about the z-axis (see Fig. 3).

At first glance, this looks like a second semester calculus problem since the E is a volume of rotation. However, we may wish to find an easier way since the calculations appears to be nontrivial.

So let $S = \partial E$. Then by (7), S can be defined by the vector equation

$$\mathbf{r}(\phi, \theta) = \sin\phi\cos\theta\,\mathbf{i} + \sin\phi\sin\theta\,\mathbf{j} + \sin 2\phi\,\mathbf{k}, \quad (\phi, \theta) \in D$$

where $D = \{(\phi, \theta) : 0 \leq \phi \leq \pi,\ 0 \leq \theta \leq 2\pi\}$.

Now

$$\mathbf{r}_\phi \times \mathbf{r}_\theta = -2\cos 2\phi\sin\phi\cot\theta\,\mathbf{i} - 2\cos 2\phi\sin\phi\sin\theta\,\mathbf{j} + \sin\phi\cos\phi\,\mathbf{k}$$

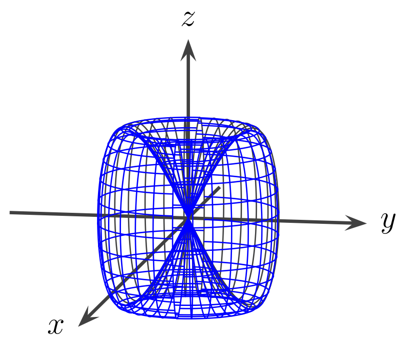

Figure 3: Volume of Rotation E

Now we need to choose a vector field $\mathbf{F}$ so that $\nabla \cdot \mathbf{F} = 1$. Let's try $\mathbf{F} = z\,\mathbf{k}$. So by the Divergence Theorem

$$\begin{aligned}
\iiint_E dV &= \iiint_E \nabla \cdot z\,\mathbf{k}\,dV \\
&= \iint_S z\,\mathbf{k} \cdot \mathbf{n}\,dS \\
&= \iint_D z\,\mathbf{k} \cdot \mathbf{r}_\phi \times \mathbf{r}_\theta\,dA \\
&= \int_0^{2\pi} \int_0^{\pi} \sin 2\phi \sin\phi \cos\phi\,d\phi\,d\theta \\
&= \pi \int_0^{\pi} \sin^2 2\phi\,d\phi \\
&= \frac{\pi}{2} \int_0^{2\pi} \sin^2 u\,du \\
&= \frac{\pi^2}{2}
\end{aligned}$$

It is a worthwhile to rework this example by choosing another vector field $\mathbf{G}$ so that $\nabla \cdot \mathbf{G} = 1$. For example, one might try $\mathbf{G} = y\,\mathbf{j}$ or $\mathbf{G} = 2x\,\mathbf{i} - z\,\mathbf{k}$.

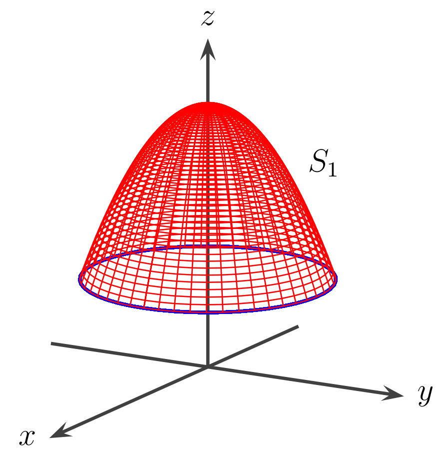

Figure 3: Parabolic Cap

Example 6. Let $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + xz\,\mathbf{k}$. Find the outward flux (away from the origin) across the parabolic cap S_1 defined by
$z = 10 - x^2 - y^2,\ z \geq 1$. That is, evaluate the flux integral $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS$.

Notice that the Divergence Theorem does not apply since S_1 is not a closed region in space. So let $S = S_1 \cup S_2$ where $S_2 : x^2 + y^2 \leq 9,\ z = 1$. Then S encloses a region E in space. So by the Divergence Theorem

$$\begin{aligned}
\iint_S \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_E \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_E (y + z + x)\, dV \\
&= \int_0^{2\pi} \int_0^3 \int_1^{10-r^2} (r\cos\theta + r\sin\theta + z) r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^3 (9 - r^2) r^3 (\cos\theta + \sin\theta)\, dr\, d\theta \\
&\qquad + \frac{1}{2} \int_0^{2\pi} \int_0^3 ((10 - r^2)^2 - 1) r\, dr\, d\theta
\end{aligned}$$

The first integral is zero and hence

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \frac{1}{2}\int_0^{2\pi}\int_0^3((10-r^2)^2-1)r\,dr\,d\theta$$

$$= \pi\int_0^3((10-r^2)^2-1)r\,dr$$

$$\vdots$$

$$= 162\pi$$

Now observe that

$$162\pi = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS$$

and we are done since the last integral is zero. Compare with the Green's Theorem handout.

Exercise: Verify the missing calculations in Example 6. In particular, evaluate $\iint_{S_2}\mathbf{F}\cdot\mathbf{n}\,dS$. Also, evaluate the flux integral $\iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS$ directly.

Example 7. Find the outward flux of $\mathbf{F}$ across the boundary of the region D if

$$\mathbf{F} = y\,\mathbf{i} + xy\,\mathbf{j} - z\,\mathbf{k}$$

and D is the region inside the solid cylinder $x^2 + y^2 \leq 4$ between the plane $z = 0$ and the paraboloid $z = x^2 + y^2$.

The boundary of D consists of the 3 surfaces (defined below). Instead of evaluating 3 surface integrals, let's invoke the Divergence Theorem. So

$$\begin{aligned}
\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\
&= \iiint_D (0 + x - 1)\, dV \\
&= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{x^2+y^2} (x-1)\, dz\, dy\, dx \\
&= \int_0^{2\pi} \int_0^2 \int_0^{r^2} (r\cos\theta - 1) r\, dz\, dr\, d\theta \\
&= \int_0^{2\pi} \int_0^2 (r\cos\theta - 1) r^3\, dr\, d\theta \\
&= \int_0^{2\pi} \left(\frac{32\cos\theta}{5} - \frac{16}{4} \right) d\theta \\
&= -8\pi
\end{aligned}$$

Let's evaluate the surface integrals directly. As we mentioned above, $\partial D = S_1 \cup S_2 \cup S_3$ where

$$S_1\colon x^2 + y^2 \le 4,\ z = 0$$
$$S_2\colon x^2 + y^2 = 4,\ 0 \le z \le 4$$
$$S_3\colon z = x^2 + y^2,\ 0 \le z \le 4$$

It is an easy exercise to see that $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

Notice that S_2 (the cylinder) can be parameterized by the vector equation

$$\mathbf{r}(s,t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + s\,\mathbf{k}, \quad (s,t) \in R$$

where $R = \{(s,t) : 0 \le t \le 2\pi,\ 0 \le s \le 4\}$.

Now

$$\mathbf{r}_s = \mathbf{k}$$
$$\mathbf{r}_t = -2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j}$$

and

$$\mathbf{r}_s \times \mathbf{r}_t = -2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j}$$

Observe that $\mathbf{n} = -(\mathbf{r}_s \times \mathbf{r}_t)$. Thus

$$\begin{aligned}
\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS &= -\int_0^{2\pi} \int_0^4 \mathbf{F}(\mathbf{r}(s,t)) \cdot (\mathbf{r}_s \times \mathbf{r}_t)\, ds\, dt \\
&= -\int_0^{2\pi} \int_0^4 (2\sin t\, \mathbf{i} + (2\cos t)(2\sin t)\, \mathbf{j} - s\, \mathbf{k}) \\
&\qquad\qquad \cdot (-2\cos t\, \mathbf{i} - 2\sin t\, \mathbf{j})\, ds\, dt \\
&= 4\int_0^{2\pi} \left(4\sin t\cos t + 8\sin^2 t\cos t\right)\, dt \\
&= 4\int_0^{2\pi} \left(4\sin t + 8\sin^2 t\right)\cos t\, dt
\end{aligned}$$

And it is now easy to see that this last integral is zero.

Finally, we compute the flux across the surface S_3 (the paraboloid). The surface can be parameterized by the vector equation

$$\mathbf{r}(s,t) = s\cos t\, \mathbf{i} + s\sin t\, \mathbf{j} + s^2\, \mathbf{k}, \quad (s,t) \in R$$

where $R = \{(s,t) : 0 \le t \le 2\pi,\ 0 \le s \le 2\}$.

So

$$\begin{aligned}
\mathbf{r}_s &= \cos t\, \mathbf{i} + \sin t\, \mathbf{j} + 2s\, \mathbf{k} \\
\mathbf{r}_t &= -s\sin t\, \mathbf{i} + s\cos t\, \mathbf{j}
\end{aligned}$$

and

$$\mathbf{r}_s \times \mathbf{r}_t = -2s^2\cos t\, \mathbf{i} - 2s^2\sin t\, \mathbf{j} + s\, \mathbf{k}$$

We also have

$$\mathbf{F}(\mathbf{r}(s,t)) = s\sin t\,\mathbf{i} + s^2\cos t\sin t\,\mathbf{j} - s^2\,\mathbf{k}$$

So that

$$\mathbf{F}(\mathbf{r}(s,t))\cdot(\mathbf{r}_s\times\mathbf{r}_t) = -s^3 - 2s^3\sin t\cos t - 2s^4\sin^2 t\cos t$$

It follows that

$$\begin{aligned}
\iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS &= \int_0^{2\pi}\int_0^2(-s^3 - 2s^3\sin t\cos t - 2s^4\sin^2 t\cos t)\,ds\,dt \\
&= \int_0^{2\pi}\int_0^2 -s^3\,ds\,dt - 2\int_0^{2\pi}\int_0^2 s^3\sin t\cos t\,ds\,dt \\
&\qquad - 2\int_0^{2\pi}\int_0^2 s^4\sin^2 t\cos t\,ds\,dt \\
&= -2\pi\int_0^2 s^3\,ds + 0 + 0 \\
&= -8\pi
\end{aligned}$$

Finally,

$$\begin{aligned}
\iint_{\partial D}\mathbf{F}\cdot\mathbf{n}\,dS &= \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_3}\mathbf{F}\cdot\mathbf{n}\,dS \\
&= -8\pi + 0 + 0
\end{aligned}$$

as expected.

Example 8. Find the outward flux of $\mathbf{F}$ across the boundary of the region D if

$$\mathbf{F} = 2xz\,\mathbf{i} + xy\,\mathbf{j} - z^2\,\mathbf{k}$$

and D is the wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$.

The surface integrals do not look particularly inviting. So once again, we try the Divergence Theorem. Now

$$\nabla \cdot \mathbf{F} = 2z + x - 2z = x$$

Thus

$$\begin{aligned}
\iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\, dS &= \iiint_D \nabla \cdot \mathbf{F}\, dV \\
&= \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} x\, dz\, dy\, dx \\
&= \int_0^2 \int_0^{\sqrt{16-4x^2}} x(4-y)\, dy\, dx \\
&= 4\int_0^2 x\sqrt{16-4x^2}\, dx - \frac{1}{2}\int_0^2 x(16-4x^2)\, dx \\
&= \frac{1}{2}\int_0^{16} \sqrt{u}\, du - \frac{1}{2}\int_0^2 (16x - 4x^3)\, dx \\
&= \frac{64}{3} - \frac{1}{2}(8x^2 - x^4)\Big|_0^2 \\
&= \frac{64}{3} - \frac{24}{3} = \frac{40}{3}
\end{aligned}$$

Let's try computing the flux directly. The boundary of D is made up of 5 distinct surfaces, which we will define along the way.

Let S_1 be the intersection of ∂D and the plane $y = 0$. Notice that the outward normal is $\mathbf{n} = -\mathbf{j}$ and since $\mathbf{F} \cdot \mathbf{n} = -xy = 0$ (for $y = 0$), we see that $\iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = 0$.

The situation is similar for the intersection of ∂D with the planes $x = 0$ and $z = 0$ (S_2 and S_3, resp.). That is,

$$\iint_{S_2} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_3} \mathbf{F} \cdot \mathbf{n}\, dS = 0$$

Now let S_4 be the intersection of ∂D with the plane $y + z = 4$. Then S_4 can be parameterized by the vector equation

$$\mathbf{r}(x, y) = x\,\mathbf{i} + y\,\mathbf{j} + (4 - y)\,\mathbf{k}, \quad (x, y) \in R$$

where $R = \{(x, y) : 0 \le x \le 2,\ 0 \le y \le \sqrt{16 - 4x^2}\}$. It follows that

$$\mathbf{r}_x = \mathbf{i} \quad \text{and} \quad \mathbf{r}_y = \mathbf{j} - \mathbf{k}$$

So that

$$\mathbf{r}_x \times \mathbf{r}_y = \mathbf{j} + \mathbf{k}$$

as expected. We also have

$$\begin{aligned} \mathbf{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y) &= \big(2xz\,\mathbf{i} + xy\,\mathbf{j} - z^2\,\mathbf{k}\big) \cdot (\mathbf{j} + \mathbf{k}) \\ &= xy - z^2 = xy - (4 - y)^2 \end{aligned}$$

It follows that

$$\iint_{S_4} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \left(xy - (4-y)^2\right)\, dA$$

$$= \int_0^2 \int_0^{\sqrt{16-4x^2}} \left(xy - (4-y)^2\right)\, dy\, dx$$

$$= \vdots$$

$$= \frac{280}{3} - 40\pi$$

Finally, let S_5 be the intersection with ∂D and the cylinder $4x^2 + y^2 = 16$. Then S_5 can be parameterized by the vector equation

$$\mathbf{r}(y, z) = \frac{\sqrt{16 - y^2}}{2}\, \mathbf{i} + y\, \mathbf{j} + z\, \mathbf{k}, \quad (y, z) \in R$$

where $R = \{(y, z) : 0 \le y \le 4,\ 0 \le z \le 4 - y\}$.

It follows that

$$\mathbf{r}_y = \frac{-y}{2\sqrt{16 - y^2}}\, \mathbf{i} + \mathbf{j}$$

$$\mathbf{r}_z = \mathbf{k}$$

and

$$\mathbf{r}_y \times \mathbf{r}_z = \mathbf{i} + \frac{y}{2\sqrt{16 - y^2}}\, \mathbf{j}$$

Now

$$\mathbf{F}\cdot(\mathbf{r}_y\times\mathbf{r}_z) = \big(2xz\,\mathbf{i} + xy\,\mathbf{j} - z^2\,\mathbf{k}\big)\cdot\left(\mathbf{i} + \frac{y}{2\sqrt{16-y^2}}\,\mathbf{j}\right)$$
$$= 2xz + \frac{xy^2}{2\sqrt{16-y^2}}$$
$$= z\sqrt{16-y^2} + \frac{y^2}{4}$$

It follows that

$$\iint_{S_5}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_R\left(z\sqrt{16-y^2}+\frac{y^2}{4}\right)\,dA$$
$$= \int_0^4\int_0^{4-y}\left(z\sqrt{16-y^2}+\frac{y^2}{4}\right)\,dz\,dy$$
$$= \vdots$$
$$= 40(\pi-2)$$

Finally,

$$\iint_{\partial D}\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\mathbf{F}\cdot\mathbf{n}\,dS + \cdots + \iint_{S_5}\mathbf{F}\cdot\mathbf{n}\,dS$$
$$= 0+0+0+\left(\frac{280}{3}-40\pi\right)+40(\pi-2)$$
$$= \frac{40}{3}$$

as we saw above!

Integral Theorems, Flux and Flow - Summary

As we saw earlier, we can imagine the **del** operator defined in this chapter as also being defined on two-dimensional vector fields by writing

$$\mathbf{F} = M\,\mathbf{i} + N\,\mathbf{j} = M\,\mathbf{i} + N\,\mathbf{j} + 0\,\mathbf{k}$$

whenever it is appropriate.

Now using the “del” notation we can rewrite all the integral theorems using a *uniform* notation.

We recall a few important definitions.

Definition. Circulation Density at a Point

The **circulation density** or **curl** of a vector field $\mathbf{F}$ is

$$\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F}$$
$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

As we saw earlier, this reduces to the usual $\mathbf{k}$-component of curl whenever $P = 0$ and $\mathbf{F} = \mathbf{F}\,\Big|_{z=0}$.

Definition. Flux Density at a Point

The **flux density** or **divergence** of a vector field $\mathbf{F}$ is

$$\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$$

Once again, this reduces to the usual two-dimensional version whenever $P = 0$.

For circulation around a smooth closed curve C we have

Green's Theorem (Tangential Form):

$$\begin{aligned}\oint_C \mathbf{F} \cdot d\mathbf{r} &= \oint_C M\,dx + N\,dy \\ &= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)\,dA \\ &= \iint_R \nabla \times \mathbf{F} \cdot\ \mathbf{k}\,dA\end{aligned}$$

Stokes' Theorem:

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\,dS$$

In the first case, C is the boundary of the plane region R. In the second, C is the boundary of the oriented surface S.

For the flux around the smooth closed curve C of an orientable surface S we have

Green's Theorem (Normal Form):

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx$$
$$= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)\,dA$$
$$= \iint_R \nabla\cdot\mathbf{F}\,dA$$

Divergence Theorem:

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_D \nabla\cdot\mathbf{F}\,dV$$

Once again C is the boundary of the plane region R and D is the region enclosed by the oriented surface S.

Some finishing touches.

Let f be a differentiable function on $I = [a, b]$ and let $\mathbf{F} = f(x)\,\mathbf{i}$ throughout I. Then $\mathbf{i}$ is the unit outward normal at the boundary point b and $-\mathbf{i}$ is the unit outward normal at a.

Thus

$$\begin{aligned} f(b) - f(a) &= (f(b)\,\mathbf{i}) \cdot \mathbf{i} + (f(a)\,\mathbf{i}) \cdot (-\mathbf{i}) \\ &= \underbrace{\mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n}}_{\substack{\text{total outward flux of } \mathbf{F} \\ \text{across the boundary of } [a,b]}} \end{aligned}$$

So by the Fundamental Theorem of Calculus

$$\begin{aligned} \mathbf{F}(b) \cdot \mathbf{n} + \mathbf{F}(a) \cdot \mathbf{n} &= f(b) - f(a) \\ &= \int_a^b f'(x)\,dx \\ &= \int_{[a,b]} \frac{\partial f}{\partial x}\,dx \\ &= \int_{[a,b]} \nabla \cdot \mathbf{F}\,dx \end{aligned}$$

In other words, the Fundamental Theorem of Calculus, the normal form of Green's Theorem, and the Divergence Theorem say that integral of the differential operator $\nabla\cdot$ operating on a field $\mathbf{F}$ over some region R (in one, two or three dimensions) is equal to the sum of the (unit) normal field components of $\mathbf{F}$ over the boundary of R.