1. (5 + 10 points) Let A be a nonempty set of real numbers.

   (a) State the definition of $\sup A$.

   Solution:

   Suppose that A is bounded (above). Then $\gamma = \sup A$ provided:

   (i*) $\gamma > a$ for all $a \in A$. That is, γ is an upper bound of A.

   (ii*) If β is any upper bound of A, then $\gamma \leq \beta$.

   If A is not bounded above, then we say that $\sup A = \infty$.

   (b) Suppose that α is a real number that satisfies the following properties:

   (i) α is an upper bound of A.

   (ii) For each $\varepsilon > 0$, there exists an $a \in A$ such that

$$\alpha - \varepsilon < a \leq \alpha$$

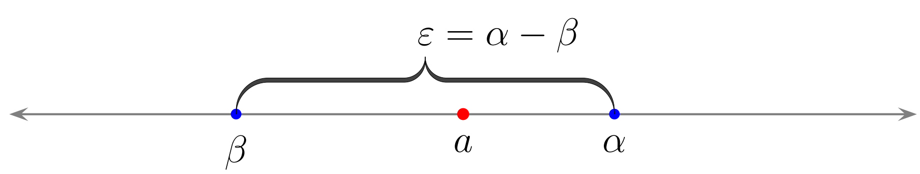
   Show that $\alpha = \sup A$. (*Note:* In class we used this *alternate* characterization to prove several important results.)

   Solution:

   Since (i) and (i*) are the same we need only show (ii) implies (ii*). So let β be any upper bound of A. If β is less than α, then we let $\varepsilon = \alpha - \beta > 0$. So by (LHS of) (ii), there exists an $a \in A$ such that

$$a > \alpha - \varepsilon = \alpha - (\alpha - \beta) = \beta$$

   so that β is not an upper bound of A (see the sketch).



2. (15 points) Suppose that A and B are bounded nonempty subsets of real numbers with $\sup A < \sup B$. Show that there is an element $b \in B$ that is an upper bound for A.

Solution:

Since there is separation between the two supremums, we should be able to find an element in $b_0 \in B$ that is arbitrarily close to $\sup B$ and also greater than $\sup A$. We give three (similar) proofs. In each case, we discover an element in B that is greater than α (the least upper bound of A).

Method 1: Let $\alpha = \sup A,\ \beta = \sup B,$ and $\varepsilon = \frac{\beta - \alpha}{2}$. Observe that, by the Axiom of Completeness, α and β are finite and that $\varepsilon > 0$. Now this choice of ε guarantees that

$$\alpha < \beta - \varepsilon$$

Also, according to the alternative characterization of the supremum, there is an element $b_0 \in B$ such that

$$\beta - \varepsilon < b_0 \ \leq \beta$$

Together these imply that

$$\alpha < b_0$$

In other words, b_0 is an upper bound for A.

Method 2: Using the same notation as above. Let μ be the midpoint between α and β. That is, let

$$\mu = \frac{\alpha + \beta}{2}$$

(*Note:* The following argument will work with any $\mu \in (\alpha, \beta)$. The midpoint is just a convenient choice.)

Now $\alpha < \mu < \beta$ and so μ is an upper bound of A. However, μ may not be an element of B. Fortunately, there is $b_0 \in B$ with $\mu < b_0 \leq \beta$. For if no such b_0 exists, then μ is an upper bound of B that is less than β, which is impossible. The result follows.

Method 3: This method is motivated by exercise 2.11.11 (or 2.10.7). Let α and β be as defined above. So according to 2.11.11, there exists a sequence $\{b_n\} \subset B$ such that $b_n \to \beta$ as $n \to \infty$. Let $\varepsilon = \beta - \alpha$. There exists an $N \in \mathbb{N}$ such that $|b_N - \beta| < \varepsilon = \beta - \alpha$. It follows that

$$\alpha - \beta = -\varepsilon < b_N - \beta < \varepsilon = \beta - \alpha$$

Now the left inequality implies that $b_N > \alpha$, as desired.

3. (15 points) Let $a_1 = 1$, and for each $n \in \mathbb{N}$ let

(1)
$$a_{n+1} = \frac{2a_n + 5}{4}$$

(a) Show that a_n is rational for each $n \in \mathbb{N}$.

Solution:

We proceed by induction on n. Clearly $a_1 = 1$ is a rational number. Now suppose that $a_n = p/q,\ p, q \in \mathbb{Z},\ q \neq 0$. Then

$$a_{n+1} = \frac{2a_n + 5}{4} = \frac{2p/q + 5}{4} = \frac{2p + 5q}{4q} \in \mathbb{Q}$$

since $2p + 5q, 4q \in \mathbb{Z}$.

(b) The sequence $\{a_n\}$ is clearly bounded below by 0. Show that it is also bounded above. (*Note:* Try showing $a_n \leq 100$, for all $n \in \mathbb{N}$.)

Solution:

We claim that a_n is bounded above, by 100. We proceed by induction on n.

$P(1)$: $a_1 = 1 \leq 100$ is obvious.

$P(n)$: Now suppose that $a_n \leq 100$ (the induction hypothesis). Then

$$\begin{aligned} 2a_n &\leq 200 \\ \implies 2a_n + 5 &\leq 205 \\ \implies a_{n+1} = \frac{2a_n + 5}{4} &\leq \frac{205}{4} < 100 \end{aligned}$$

as desired. Observe that the exact same argument will also show that the sequence is bounded above by 5/2 (see below).

(c) *Carefully* prove that the sequence $\{a_n\}$ converges, and find its limit. *Be sure to specify any theorems that you use.*

Solution:

i. Suppose first that the sequence converges, say $\lim_{n\to\infty} a_n = s$. Then by (1) we have

$$\begin{aligned} s = \lim_{n\to\infty} a_{n+1} &= \lim_{n\to\infty} \frac{2a_n + 5}{4} \\ &= \frac{2s+5}{4} \qquad \text{(justified by the Limit Laws)} \end{aligned}$$

Solving for s we obtain

$$s = 5/2$$

ii. We claim that $\{a_n\}$ is increasing. For $n \geq 1$, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{2a_n + 5}{4} - \frac{4a_n}{4} \\ &= \frac{5 - a_n}{4} > 0 \end{aligned}$$

since $a_n < 5/2$ for all $n \in \mathbb{N}$.

Hence, the sequence converges by the Monotone Convergence Theorem, and the first step has been justified.

4. (15 points) Use an ε-N argument to prove

$$\lim_{n\to\infty} \frac{3n^2+6}{n^2-3n} = 3$$

Solution:

We omit the "scrap work". Let $\varepsilon > 0$ and let $N = 15/\varepsilon + 3$. We remark that with this choice, $N > 3$. Now

$$\begin{aligned} n > N &> \frac{15}{\varepsilon} + 3 \\ \implies \varepsilon &> \frac{15}{n-3} = \frac{15n}{n(n-3)} \\ &\geq \frac{6+9n}{n^2-3n} = \left|\frac{6+9n}{n^2-3n}\right| \quad \text{(since } n > 3) \\ &= \left|\frac{3n^2+6}{n^2-3n} - 3\right| \end{aligned}$$

In other words

$$\left|\frac{3n^2+6}{n^2-3n} - 3\right| < \varepsilon$$

as desired.

5. (15 points) Let $b > 1$.

   (a) Prove that $b^{n+1} > b^n$ for all $n \in \mathbb{N}$.

   Solution:

   Claim that $b > 0 \implies b^n > 0$ for all $n \in \mathbb{N}$. If the claim is true, then

   $$b^n \cdot b > b^n \cdot 1$$

   by Theorem 3.2. To prove the claim, we proceed by induction. Clearly $b^1 > 0$. Now suppose that $b^n > 0$. Then by Theorem 3.2, $b \cdot b^n = b^{n+1} > 0$.

(b) Prove that $b^n - 1 \geq n(b-1)$ for all $n \in \mathbb{N}$.

Solution:

We give 2 proofs.

Method 1: We proceed by induction on n. For $n = 1$ we have

$$b^1 - 1 = 1(b-1) \quad \checkmark$$

Now suppose that $b^n - 1 \geq n(b-1)$ holds. Notice that by part (a), $b^n > 1$ hence $b^n(b-1) > 1(b-1)$. Thus

$$\begin{aligned} b^{n+1} - 1 &= (b^{n+1} - b^n) + (b^n - 1) && \text{(arithmetic)} \\ &\geq (b^{n+1} - b^n) + n(b-1) && \text{(by the induction hypothesis)} \\ &= b^n(b-1) + n(b-1) && \text{(arithmetic)} \\ &> (b-1) + n(b-1) && \text{(see comments above)} \\ &= (n+1)(b-1) && \text{(arithmetic)} \end{aligned}$$

as desired.

Method 2: By part (a), $b^k > 1$ for all $k \in \mathbb{N}$. We motivate this approach by observing that we have equality for $n = 1$. For $n = 2$, we have

$$b^2 - 1 = (b+1)(b-1) > (1+1)(b-1) \text{ (since } b > 1)$$

That's encouraging. Now recall that $b^n - 1$ has a well-known factorization. Thus

$$\begin{aligned} b^n - 1 &= \underbrace{(b^{n-1} + b^{n-2} + \cdots + b + 1)}_{n \text{ terms}}(b-1) \\ &> \underbrace{(1 + 1 + \cdots + 1 + 1)}_{n \text{ terms}}(b-1), \qquad \text{(by part (a))} \\ &= n(b-1) \end{aligned}$$

as desired.

(c) Suppose that $0 < a < 1$. Prove that $a < \sqrt{a} < 1$. (*Note:* By definition, $\sqrt{a} > 0$.)

Solution:

We claim that if $0 < a < 1$ then $\sqrt{a} < 1$. If the claim is true then the result follows by Theorem 3.2 since

$$0 < \sqrt{a} \quad \Longrightarrow \quad a = \sqrt{a} \cdot \sqrt{a} < \sqrt{a} \cdot 1 < 1$$

Now suppose the claim is false, i.e., suppose that $\sqrt{a} \geq 1$. Then by part (a),

$$1 \leq \sqrt{a} \leq \left(\sqrt{a}\right)^2 = a$$

contrary to the given conditions on a.

6. (10 points) Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences and $M > 0$. If $\lim_{n\to\infty} a_n = 0$ and $-M \leq b_n \leq M$ for all $n \in \mathbb{N}$, use an ε-N argument to prove

$$\lim_{n\to\infty} a_n b_n = 0$$

Solution:

From the Limit theorems, we see that

$$\lim_{n\to\infty} \pm M a_n = \pm M \lim_{n\to\infty} a_n = 0$$

Now since

$$-M|a_n| \leq b_n|a_n| \leq M|a_n|, \quad \forall n \in \mathbb{N}$$

The desired result follows from the Squeeze Law (see Exercise 2.3.3 in the text). However, we were asked to prove this result using an ε-N argument. Nevertheless, the Squeeze Law provides some intuition.

Now let $\varepsilon > 0$. Since $\lim_{n\to\infty} a_n = 0$, we can choose an $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - 0| < \varepsilon/M$. It follows that

$$|a_n b_n - 0| = |a_n b_n| = |a_n|\,|b_n| \leq |a_n| M \leq \frac{\varepsilon}{M}\, M$$

provided that $n \geq N$.

7. (15 points) Let A and B be bounded sets and define $A+B=\{a+b : a\in A \text{ and } b\in B\}$. Show that $\inf A+\inf B=\inf(A+B)$.

   HINT: Let $\alpha=\inf A$ and $\beta=\inf B$. First show that $\alpha+\beta$ is a lower bound of the set $A+B$.

   Solution:

   Following the hint, we let $\alpha=\inf A$ and $\beta=\inf B$. For $a\in A$ and $b\in B$, we have $a\ge\alpha$ and $b\ge\beta$. It follows that $a+b\ge\alpha+\beta$ for all $a\in A$ and $b\in B$. Hence, $\alpha+\beta$ is an lower bound for the set $A+B$. It follows by the AoC that $t=\inf(A+B)$ exists and hence

   $$t\ge\alpha+\beta \tag{2}$$

   We provide three proofs that we actually have equality.

   Method 1 (Direct): By the remarks above, we see that item (i) (see Problem 1) holds since $\alpha+\beta$ is a lower bound. Now let $\varepsilon>0$. Since $\alpha=\inf A$, there exists an $a_0\in A$ such that $\alpha\le a_0<\alpha+\varepsilon$. Similarly, there is a $b_0\in B$ such that $\beta\le b_0<\beta+\varepsilon$. Adding these inequalities together, we have shown that for an arbitrary $\varepsilon>0$, we can find an element $t_0=a_0+b_0\in A+B$ such that

   $$\alpha+\beta\le a_0+b_0<\alpha+\beta+2\varepsilon.$$

   which is item (ii).

   Method 2 (Contrapositive): Suppose we have a strict inequality in (2). That is, suppose $t>\alpha+\beta$ and set $\varepsilon=t-(\alpha+\beta)>0$. So by the alternate characterization of the infimum, there is $a_0\in A$ and $b_0\in B$ such that

   $$\alpha\le a_0<\alpha+\varepsilon/2$$
   $$\beta\le b_0<\beta+\varepsilon/2$$

   Adding these together, we obtain

   $$\begin{aligned} a_0+b_0 &< \alpha+\beta+\varepsilon\\ &= \alpha+\beta+(t-\alpha-\beta)\\ &= t \end{aligned}$$

   So t is not an lower bound of $A+B$ contrary to our definition. The result follows.

Method 3: Choose an arbitrary but fixed element $b_0 \in B$. Since t is the infimum of $A+B$, we know that $a + b_0 \geq t$ for all $a \in A$. In particular, $a \geq t - b_0$ for all $a \in A$. Hence $t - b_0$ is an lower bound of A.

Since α is the greatest lower bound of A, we see that $\alpha \geq t - b_0$. Rearranging this last inequality we see that $b_0 \geq t - \alpha$. Since b_0 was arbitrary, we conclude that $b \geq t - \alpha$ for all $b \in B$ and hence, $t - \alpha$ is a lower bound for B. It follows that $\beta \geq t - \alpha$. In other words,

$$t \leq \alpha + \beta \tag{3}$$

Together (2) and (3) imply the result.

8. (**Bonus** - 10 points) Prove that

$$\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$$

Solution:

We omit the "scrap work". Let $\varepsilon > 0$ and choose $N = \frac{1}{\varepsilon^2}$. Now for any $n > N$ we have

$$\begin{aligned}
\left|\sqrt{n+1} - \sqrt{n}\right| &= \sqrt{n+1} - \sqrt{n} \\
&= \frac{\sqrt{n+1} - \sqrt{n}}{1} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\
&= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\
&< \frac{1}{2\sqrt{n}} < \frac{1}{\sqrt{n}} < \varepsilon
\end{aligned}$$

as desired.