1. (10 points) Prove that for all $n \geq 2$, $\dfrac{1}{\sqrt{1}} + \dfrac{1}{\sqrt{2}} + \cdots + \dfrac{1}{\sqrt{n}} > \sqrt{n}$. (*Hint:* To show the base case ($n = 2$), you must show that $1 + 1/\sqrt{2} > \sqrt{2}$. Recall that if a, b are positive, then $a > b$ iff $a^2 > b^2$. So try showing that $a^2 - b^2 > 0$ for the appropriate choice of a and b.)

Solution:

(C.f.- Bonus problem on Exam 1.) For the base case we follow the hint (twice).

$$\left(1 + \frac{1}{\sqrt{2}}\right)^2 - \left(\sqrt{2}\right)^2 = \sqrt{2} - \frac{1}{2} > 0$$

since $\sqrt{2} > 1$. Now suppose that $\sum_{j=1}^{n} \frac{1}{\sqrt{j}} > \sqrt{n}$. Then

$$\begin{aligned} \sum_{j=1}^{n+1} \frac{1}{\sqrt{j}} &= \sum_{j=1}^{n} \frac{1}{\sqrt{j}} + \frac{1}{\sqrt{n+1}} \\ &> \sqrt{n} + \frac{1}{\sqrt{n+1}} \\ &= \frac{\sqrt{n(n+1)} + 1}{\sqrt{n+1}} \\ &> \frac{n+1}{\sqrt{n+1}} = \sqrt{n+1} \end{aligned}$$

2. (10 points) Use an ε-N argument to prove

$$\lim_{n \to \infty} \frac{5n^2 - n}{n^2 + 2} = 5$$

Solution:

We omit the "scrap work". Let $\varepsilon > 0$ and let $N = 2/\varepsilon + 10$. We remark that with this choice, $N > 10$. Now

$$\begin{aligned} n > N > \frac{2}{\varepsilon} + 10 &> \frac{2}{\varepsilon} \\ \Longrightarrow \varepsilon &> \frac{2}{n} = \frac{2n}{n^2} \\ &\geq \frac{2n}{n^2+2} > \frac{n+10}{n^2+2} \qquad \text{(since } n > 10\text{)} \\ &= \left| \frac{n+10}{n^2+2} \right| = \left| \frac{-n-10}{n^2+2} \right| \\ &= \left| \frac{5n^2 - n}{n^2+2} - 5 \right| \end{aligned}$$

In other words, $n > 2/\varepsilon + 10$ implies that

$$\left| \frac{5n^2 - n}{n^2+2} - 5 \right| < \varepsilon$$

as desired.

3. (10 points) Suppose that $\{a_n\}$ is a convergent sequence, say $\lim_{n\to\infty} a_n = a$. If $a_n \leq b$ for all $n \in \mathbb{N}$, prove the following:

$$\text{(i)} \quad a \leq b$$

$$\text{(ii)} \quad a \leq \sup_n a_n$$

Solution:

If the conclusion in (i) is false then $\frac{c-b}{2} > 0$ and there exists an $N \in \mathbb{N}$ such that

$$\frac{b-c}{2} < a_N - c < \text{Who cares!}$$

Rearranging yields

$$a_N > \frac{b+c}{2} > b$$

contrary to our assumptions. Since this result holds for any upper bound b, (ii) follows immediately.