Throughout this exam you may assume that $A \subseteq \mathbb{R}$ is never the empty set.

1. (10 points) Let $\{a_n\}$ and $\{b_n\}$ be bounded sequences of positive real numbers. If $\sum_{n=1}^{\infty} a_n$ converges show that $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

Solution:

We use the Comparison test. Since $\{b_n\}$ is bounded there is an $L > 0$ such that for all $n \in \mathbb{N}$ we have $0 < b_n < L$. Now $a_n > 0$ implies

$$0 < a_n b_n < La_n, \quad n \in \mathbb{N} \tag{1}$$

By the Algebraic Limit theorems,

$$\sum_{n=1}^{\infty} La_n = L \sum_{n=1}^{\infty} a_n < \infty \tag{2}$$

The result now follows by combining (1) and (2) and invoking the Comparison test.

Remark. Here's another approach. Note the similarities to the proof above. Let $\varepsilon > 0$. Then by the Cauchy Criterion for series, there exists $N \in \mathbb{N}$ such that for all $n, m > N$ we have

$$\left| \sum_{j=m}^{n} a_j \right| = \sum_{j=m}^{n} a_j < \frac{\varepsilon}{L}$$

Thus

$$\begin{aligned} \left| \sum_{j=m}^{n} a_j b_j \right| = \sum_{j=m}^{n} a_j b_j &\le L \sum_{j=m}^{n} a_j \\ &< L\, \frac{\varepsilon}{L} \end{aligned}$$

The desired result now follows by the Cauchy Criterion for series.

2. (15 points) Use an ε-δ argument to prove that $f(x) = 5x^2 + 1$ is continuous at 3.

Solution:

We need to show that $\lim_{x \to -2} x^2 = 4$. There are several ways to prove this. For the standard approach, observe that if $|x - 3| < 1$, then $|x + 3|$ is bounded above by 7.

Now let $\varepsilon > 0$ and choose $\delta = \min\{1, \varepsilon/35\}$. Then $|x - 3| < \delta$ implies

$$\begin{aligned} |f(x) - f(3)| &= |5x^2 - 1 - 46| \\ &= 5|x + 3||x - 3| \\ &< 35|x - 3| < 35\,\frac{\varepsilon}{35} \end{aligned}$$

as desired.

Here's another approach. First confirm that

(3) $$5x^2 = 5(x - 3)^2 + 30(x - 3) + 45$$

and use this equality to obtain the result. Here are the details.

Let $\varepsilon > 0$. Now choose $\delta = \min\{1, \varepsilon/35\}$. Then $|x - 3| < \delta$ implies

$$\begin{aligned} |f(x) - f(3)| &= |5x^2 + 1 - 46| \\ &= |5(x - 3)^2 + 30(x - 3)| \\ &\le 5|x - 3|^2 + 30|x - 3| \\ &< 5|x - 3| + 30|x - 3|, \quad \text{(since } |x - 3| < 1\text{)} \\ &< \varepsilon \end{aligned}$$

3. (15 points)

   (a) *Carefully* state the definition of uniform continuity.

   Definition: A function $f : A \to \mathbb{R}$ is uniformly continuous on A if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $x, y \in A$ with

$$|x - y| < \delta \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon \tag{4}$$

   (b) Let $f : A \to \mathbb{R}$ be uniformly continuous. Show that if $\{x_n\} \subset A$ is a Cauchy sequence, then $\{f(x_n)\}$ is also a Cauchy sequence.

   Solution:

   Let $\varepsilon > 0$ and choose $\delta > 0$ so that (4) holds. Since $\{x_n\}$ is Cauchy, there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|x_n - x_m| < \delta$. Then by (4)

$$|f(x_n) - f(x_m)| < \varepsilon$$

   and hence, $\big\{f(x_n)\big\}$ is Cauchy.

4. (15 points) Find the interval and radius of convergence for the power series below. Don't forget to specify the behavior at the end points.

$$\sum_{n=0}^{\infty} \frac{2^n}{n} x^n$$

   Solution:

   We try the (absolute) ratio test.

$$\rho = \lim_{n\to\infty} \frac{n2^{n+1}}{(n+1)2^n} \frac{|x|^{n+1}}{|x|^n} = 2|x| \lim_{n\to\infty} \frac{n}{n+1} = 2|x| <^? 1$$

   It follows that $R = 1/2$. The given series diverges at $x = 1/2$ since the substitution $x \to 1/2$ yields the Harmonic series which is known to diverge (as was shown in class). On the other hand, the substitution $x \to -1/2$ yields the alternating Harmonic series, which is known to converge. It follows that the interval of convergence is $[-1/2, 1/2)$.

Part II - Take Home

Student Name: ________________________

1. (15 points) Let $I = [a, b]$ be a closed bounded interval and let $f : I \to \mathbb{R}$ be a continuous function. If $K = \{x \in I : f(x) \geq 0\}$ is nonempty show that K is compact (i.e., closed and bounded).

Solution:

Since I is bounded and $K \subset I$, we need only show that K is closed and then apply Theorem 3.19.7 (from the lecture notes). Let c be a limit point of K and choose $\{x_n\} \subset K\ (\subset I)$ so that $x_n \to c$ as $n \to \infty$. Now we would like to say that $f(x_n) \to f(c)$ as $n \to \infty$. However, that is only guaranteed if f is continuous at c.

Fortunately, we notice that c is also limit point of I since $K \subseteq I$. Hence $c \in I$ since I is closed. It follows that f is continuous at c. Now for each $n \in \mathbb{N}$, $x_n \in K$ implies that $f(x_n) \geq 0$. Hence

$$f(c) = \lim_{n \to \infty} f(x_n) \geq 0$$

by the Order Limit theorems from chapter 2. It follows that $c \in K$. Thus K is closed.

2. (10 points) Suppose that $f, g : A \to \mathbb{R}$ are continuous at $c \in A$ and let $h(x) = \max\{f(x), g(x)\}$. Use an ε-δ argument to show that h is continuous at c.

   Hint: You may freely use the fact that $\max\{a, b\} = \frac{1}{2}(a + b + |a - b|)$ for all $a, b \in \mathbb{R}$.

   Solution:

   Following the hint, we see that

$$h(x) = \frac{1}{2}(\underbrace{f(x) + g(x)}_{h_1(x)} + |\underbrace{f(x) - g(x)}_{h_2(x)}|)$$

   Now h_1 and h_2 are, respectively, the sum and difference of two continuous functions, hence both are continuous. Also, $|h_2(x)|$ is the composition of two continuous functions, hence it is continuous, and consequently, so is $h_1(x) + |h_2(x)|$. Finally, the product of two continuous functions ($1/2$ and $h_1(x) + |h_2(x)|$) is continuous, as desired.

   However, the instructions explicitly asked for an ε-δ argument. So let $\varepsilon > 0$ and choose $\delta_1, \delta_2 > 0$ so that $|x - c| < \delta_1$ implies $|f(x) - f(c)| < \varepsilon/2$ and $|x - c| < \delta_2$ implies $|g(x) - g(c)| < \varepsilon/2$. Now let $\delta = \min\{\delta_1, \delta_2\}$. Then $|x - c| < \delta$ implies

$$\begin{aligned}
2|h(x) - h(c)| &= \big|f(x) + g(x) + |f(x) - g(x)| - f(c) - g(c) - |f(c) - g(c)|\big| \\
&= \big|f(x) - f(c) + g(x) - g(c) + |f(x) - g(x)| - |f(c) - g(c)|\big| \\
&\leq |f(x) - f(c)| + |g(x) - g(c)| + \big||f(x) - g(x)| - |f(c) - g(c)|\big| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \big|(f(x) - g(x)) - (f(c) - g(c))\big|^* \\
&\leq \varepsilon + |f(x) - f(c)| + |g(c) - g(x)| \\
&\vdots \\
&\leq 2\varepsilon
\end{aligned}$$

   * - Since $\big||a| - |b|\big| \leq |a - b|$

   Now divide by 2 to obtain the desired result.

3. (10 points) Let $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1 - x, & \text{otherwise.} \end{cases}$ Show that f is continuous only at $x = 1/2$. (*Hint:* A sketch may help.)

Solution:

Let $\varepsilon > 0$ and choose $\delta = \varepsilon$. Then if $|x - 1/2| < \delta = \varepsilon$ we have two cases:

$$|f(x) - f(1/2)| = |x - 1/2| < \varepsilon \text{ for } x \in \mathbb{Q}$$
$$|f(x) - f(1/2)| = |(1 - x) - 1/2| = |1/2 - x| = |x - 1/2| < \varepsilon \text{ otherwise.}$$

It follows that f is continuous at $1/2$.

Now if $c \neq 1/2$, then we may choose $\varepsilon = \frac{|1-2c|}{2} = |1/2 - c|$.

If $c \in \mathbb{Q}$, then for every $\delta > 0$ we can find an irrational number $x_{\mathrm{i}} \in (c - \delta, c + \delta)$ such that

$$\begin{aligned} |f(x_{\mathrm{i}}) - f(c)| &= |(1 - x_{\mathrm{i}}) - c| \\ &> |1/2 - c| = \varepsilon \end{aligned}$$

On the other hand, if $c \notin \mathbb{Q}$ then we can find a rational number $x_{\mathrm{r}} \in (c - \delta, c + \delta)$ such that

$$\begin{aligned} |f(x_{\mathrm{r}}) - f(c)| &= |x_{\mathrm{r}} - (1 - c)| \\ &= |(1 - x_{\mathrm{r}}) - c| \\ &> |1/2 - c| = \varepsilon \end{aligned}$$

It follows that f is discontinuous at c.

4. (10 points) Show that $f(x) = \sqrt{x}\,\sin(\cos x)$ is uniformly continuous on the interval $[0, \pi)$.

Solution:

Note that $\sqrt{x}$ and $\cos x$ are continuous for all $x \geq 0$. Also, $\sin x$ is continuous on the interval $[-1, 1] = \mathrm{Ran}(\cos x)$. Since the composition of continuous functions is continuous (Theorem 3.17.5 from the text), $\sin(\cos x)$ is continuous on $[0, \pi]$. Now by Theorem 3.17.4 (text), the product of continuous functions is continuous. We have shown that f is continuous on the closed interval $[0, \pi]$. The result now follows by Theorem 3.19.5 (text).

5. (**Bonus** - 5 points) Regarding Problem 3b: Show that the uniform continuity condition can not be relaxed. That is, find a continuous function $f : A \to \mathbb{R}$ and a Cauchy sequence $\{x_n\} \subset A$ such that $\{f(x_n)\}$ is **not** a Cauchy sequence.

Solution:

Let $f(x) = 1/x$. Then f is continuous on $(0, 1]$. Now let $x_n = 1/n$. Then $x_n \to 0$ as $n \to \infty$ but $f(x_n) \to \infty$.

6. (**Double Bonus** - 5 points) The solution above used the unboundedness of f to obtain the desired result. Find a *bounded* continuous function $f : A \to \mathbb{R}$ and a Cauchy sequence $\{x_n\} \subset A$ such that $\{f(x_n)\}$ is **not** a Cauchy sequence. This is due by Thursday at 12:00 pm.