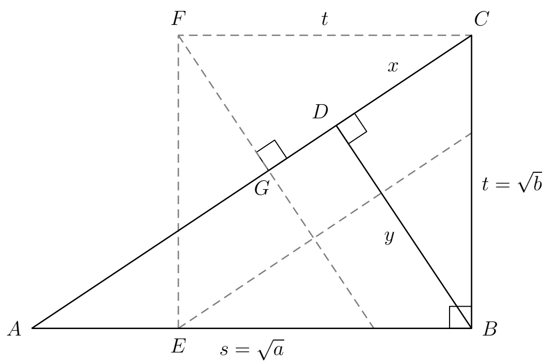
In its most basic form, the **Arithmetic-Geometric Mean Inequality (GA)** states the following. Let $a \geq b > 0$. Then

(1) $$\sqrt{ab} \leq \frac{a+b}{2}$$

with equality if and only if $a = b$.

We will investigate generalizations of the **GA** inequality along with several interesting proofs. We begin with a geometric proof.

Proof. Let $a \geq b > 0$ and let $s = \sqrt{a}, t = \sqrt{b}$. We construct the right triangle $\triangle ABC$ and the square $\square BCFE$ as shown in the sketch below.



Notice that $\triangle ABC \sim \triangle BDC \cong \triangle CGF$ and that the square contains four triangles that are congruent to $\triangle BDC$. Now by the similarity of the first two triangles, we have

(2) $$\frac{s}{y} = \frac{t}{x} \quad \text{or} \quad s = \frac{ty}{x}$$

Now since $y \geq x$, the sum of the areas of the four triangles congruent to $\triangle BDC$ is not greater than the area of the square $\square BCFE$. That is,

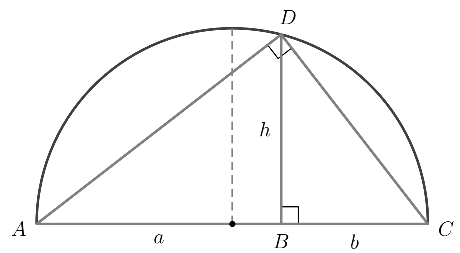
(3) $$2xy \leq t^2 = x^2 + y^2$$

Thus

$$\begin{aligned}\sqrt{ab} = st = \frac{ty}{x}\,t = xy\,\frac{t^2}{x^2} &= xy\left(1 + \frac{y^2}{x^2}\right)\\ &\leq \frac{t^2}{2}\left(1 + \frac{y^2}{x^2}\right) = \frac{1}{2}\left(t^2 + \frac{t^2y^2}{x^2}\right)\\ &= \frac{t^2 + s^2}{2}\\ &= \frac{a+b}{2}\end{aligned}$$

$\square$

Here is an easier approach. Once again, let $a \geq b > 0$ and consider a circle of diameter $a + b$ as shown in the sketch below. *Note:* $a = AB,\ b = BC$, etc.



Clearly,

$$h \leq \text{radius} = \frac{a+b}{2}$$

So it's enough to prove that $h = \sqrt{ab}$. Once again, from elementary geometry, we have

$$\triangle ABD \sim \triangle DBC$$

So by similarity

$$\frac{h}{a} = \frac{b}{h}$$

and the result is immediate.

First generalization. For $j \in \mathbb{N}$ let $a_j > 0$. Prove that

$$(a_1 a_2 \cdots a_n)^{1/n} \le \frac{1}{n}\sum_{j=1}^{n} a_j \tag{4}$$

Proof. Since (4) is trivial for $n = 1$, we will start with $n = 2$. Then we'll use induction to prove the case $n = 2^k$. Finally, we'll prove the general result.

Case 1. Observe that for $A, B > 0$ we have

$$\begin{aligned} (A - B)^2 &\ge 0 \\ \Longrightarrow A^2 - 2AB + B^2 &\ge 0 \end{aligned} \tag{5}$$

Rearranging yields

$$AB \le \frac{1}{2}(A^2 + B^2)$$

Letting $A = \sqrt{a}$ and $B = \sqrt{b}$ we obtain (4) for the special case $n = 2^1$. That is,

$$\sqrt{ab} \le \frac{a+b}{2}$$

Of course, we could have also appealed to the geometric proof above to obtain this result. Notice that because of (5), we have equality in (4) if and only if $a = b$.

Case 2. Now suppose $n = 2^k$ in (4). We proceed by induction on k. The case $k = 1$ was proven above. Now suppose that (4) holds with $n = 2^k$. We need to show that

$$(a_1 a_2 \cdots a_{2^{k+1}})^{1/2^{k+1}} \le \frac{1}{2^{k+1}}\sum_{j=1}^{2^{k+1}} a_j \tag{6}$$

Notice that there are twice as many factors on the left-hand side of (6) as there are on the left-hand side of (4).

$$\begin{aligned} (a_1 a_2 \cdots a_{2^{k+1}})^{1/2^{k+1}} &= \left(\sqrt{a_1 a_2}\sqrt{a_3 a_4}\cdots\sqrt{a_{2^{k+1}-1} a_{2^{k+1}}}\right)^{1/2^k} \\ &\le \frac{1}{2^k}\sum_{j=1}^{2^k}\sqrt{a_{2j-1}a_{2j}} && \text{(by the induction hypothesis)} \\ &\le \frac{1}{2^k}\sum_{j=1}^{2^k}\frac{1}{2}\left(a_{2j-1} + a_{2j}\right) && \text{(by repeated applications of Case 1)} \\ &\le \frac{1}{2^{k+1}}\sum_{j=1}^{2^{k+1}} a_j \end{aligned}$$

Case 3. Now for $2^{k-1} < n < 2^k$, let $m = 2^k - n$. Then

$$\begin{aligned} p = (a_1 a_2 \cdots a_n)^{1/n} &= \left\{(a_1 a_2 \cdots a_n)^{1/n}\right\}^{\frac{2^k}{2^k}} \\ &= \left\{(a_1 a_2 \cdots a_n)^{2^k/n}\right\}^{\frac{1}{2^k}} \\ &= \left((a_1 a_2 \cdots a_n)\underbrace{p \cdot p \cdots p}_{m \text{ factors}}\right)^{\frac{1}{2^k}} \\ &= \left(\underbrace{a_1 a_2 \cdots a_n \cdot p \cdot p \cdots p}_{2^k \text{ factors}}\right)^{\frac{1}{2^k}} \\ &\le \frac{1}{2^k}\left(\sum_{j=1}^{n} a_j + m \cdot p\right) \end{aligned}$$

by Case 2. Rearranging we obtain

$$\begin{aligned} p\left(1 - \frac{m}{2^k}\right) &\le \frac{1}{2^k}\sum_{j=1}^{n} a_j \\ \Longrightarrow (a_1 a_2 \cdots a_n)^{1/n} &\le \frac{2^k}{2^k - m}\frac{1}{2^k}\sum_{j=1}^{n} a_j \\ &= \frac{1}{n}\sum_{j=1}^{n} a_j \end{aligned}$$

□

Allowing Real Exponents - Now suppose that $p_j > 0$ for $j = 1, 2, \dots n$ and $\sum_{j=1}^{n} p_j = 1$. Show that

$$a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \le \sum_{j=1}^{n} p_j a_j \tag{7}$$

Remark: Observe that (4) is a special case of (7) with $p_j = 1/n,\ j = 1, 2, \dots n$.

First we prove another generalization of (3). Suppose that $p, q > 1$ are **conjugate exponents**. That is, suppose that

$$\frac{1}{p} + \frac{1}{q} = 1 \tag{8}$$

Now let $A, B > 0$. Prove that

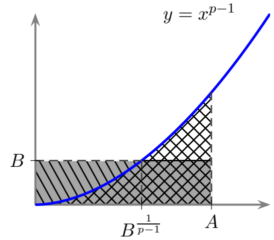
$$AB \le \frac{A^p}{p} + \frac{B^q}{q} \tag{9}$$

This is **Young's Inequality**.

Proof.

From the sketch we observe that the area of both *hatched* regions is not less than the area of a rectangle with side lengths A and B. Also, because of (8) we have

$$1 + \frac{1}{p-1} = \frac{p}{p-1} = q$$



Thus

$$\begin{aligned} AB &\le \underbrace{\int_0^A x^{p-1}\,dx}_{\text{cross-hatched area}} + \underbrace{\left(B \times B^{\frac{1}{p-1}} - \int_0^{B^{\frac{1}{p-1}}} x^{p-1}\,dx \right)}_{\text{left-hatched area}} \\ &= \frac{A^p}{p} + B^q - \frac{B^q}{p} \\ &= \frac{A^p}{p} + \left(1 - \frac{1}{p}\right) B^q \end{aligned}$$

which is (9). □

Remark. It is also clear from the sketch that we have equality precisely when $A = B^{\frac{1}{p-1}}$. That is,

$$AB = \frac{A^p}{p} + \frac{B^q}{q} \iff A^p = B^q$$

Proof. (of main result) First, suppose that $a, b > 0$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Now write $p = 1/\alpha$ and $q = 1/\beta$ and let $A = a^\alpha = a^{1/p}$ and $B = b^\beta = b^{1/q}$ in (9) to obtain

$$a^\alpha b^\beta \le \alpha a + \beta b \tag{10}$$

which is (7) for $n = 2$.

Next, suppose that $a, b, c > 0$ and $\alpha, \beta, \gamma > 0$ with $\alpha + \beta + \gamma = 1$. Then

$$\begin{aligned} a^\alpha b^\beta c^\gamma &= \left(a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\alpha}{\alpha+\beta}} \right)^{\alpha+\beta} c^\gamma \\ &\le (\alpha+\beta) \left(a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\alpha}{\alpha+\beta}} \right) + \gamma c \\ &\le (\alpha+\beta) \left(\frac{\alpha}{\alpha+\beta} a + \frac{\alpha}{\alpha+\beta} b \right) + \gamma c \\ &= \alpha a + \beta b + \gamma c \end{aligned}$$

Here we have applied the inequality (10) twice.

The path to (7) is now clear. We proceed by induction on n.

$P(1)$: $a_1^1 = 1 \cdot a_1$ is obvious.

$P(n)$: Now suppose that (7) is true (the induction hypothesis) and that $p_j > 0$ for $j = 1, 2, \dots n+1$ with $\sum_{j=1}^{n+1} p_j = 1$. Also, let $\alpha = \sum_{j=1}^{n} p_j$. Then $\alpha + p_{n+1} = 1$ and

$$\begin{aligned} a_1^{p_1} a_2^{p_2} \cdots a_{n+1}^{p_{n+1}} &= \left(a_1^{p_1/\alpha} a_2^{p_2/\alpha} \cdots a_n^{p_n/\alpha} \right)^\alpha a_{n+1}^{p_{n+1}} \\ &\le \alpha \left(a_1^{p_1/\alpha} a_2^{p_2/\alpha} \cdots a_n^{p_n/\alpha} \right) + p_{n+1} a_{n+1} \end{aligned}$$

by (10). Applying the induction hypothesis to the parenthetical quantity yields

$$\begin{aligned} &\le \alpha \left\{ (p_1/\alpha) a_1 + (p_2/\alpha) a_2 + \cdots + (p_n/\alpha) a_n \right\} + p_{n+1} a_{n+1} \\ &= \sum_{j=1}^{n} p_j a_j + p_{n+1} a_{n+1} \end{aligned}$$

as desired.

□

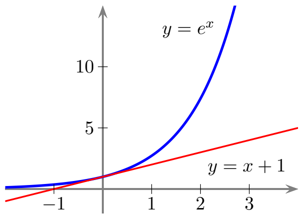
Pólya's Proof - According to J. Michael Steele, George Pólya discovered the proof outlined below in a dream.

Once again, we suppose $p_j > 0$ for $j = 1, 2, \ldots n$ and $\sum_{j=1}^n p_j = 1$. Also, let A and G denote, respectively, the arithmetic and geometric means of $a_1, a_2, \ldots, a_n$ with each $a_j > 0$. We wish to show that

$$G \le A$$

Pólya's proof begins with the observation

$$1 + x \le e^x$$



Now the substitution $x \to x - 1$ yields

$$x \le e^{x-1} \tag{11}$$

So for $j = 1, 2, \ldots, n$ the bound in (11) implies

$$a_j \le e^{a_j - 1} \text{ and } a_j^{p_j} \le e^{p_j a_j - p_j}$$

Thus

$$\begin{aligned} G = a_1^{p_1} a_2^{p_2} \cdots a_n^{p_n} \stackrel{\text{def}}{=} \prod_{j=1}^n a_j^{p_j} &\le \prod_{j=1}^n e^{p_j a_j - p_j} \\ &= e^{\sum_{j=1}^n p_j a_j - \sum_{j=1}^n p_j} \\ &= e^{A-1} \end{aligned} \tag{12}$$

Appealing to (11) once again, we see that

$$A \le e^{A-1} \tag{13}$$

And now we appear to be in trouble since (12) and (13) show that A and G are bounded above by e^{A-1}. This is not what we want. Notice however that the right-hand side of (12) is one precisely when the arithmetic mean is one.

To see how this helps us, let $\alpha_j = a_j / A$. Then

$$\sum_{j=1}^n p_j \alpha_j = \sum_{j=1}^n \frac{p_j a_j}{A} = \frac{1}{A} \sum_{j=1}^n p_j a_j = 1$$

Thus

$$\frac{1}{A} \prod_{j=1}^n a_j^{p_j} = \prod_{j=1}^n \left(\frac{a_j}{A} \right)^{p_j} = \prod_{j=1}^n \alpha_j^{p_j} \le 1 \tag{14}$$

Thus

$$\prod_{j=1}^n a_j^{p_j} \le A = \sum_{j=1}^n p_j a_j \tag{15}$$

as desired.