Let $a > 0$ and $b > 0$. Use an ε-N argument to prove the limit below. *Note:* This is one of the more annoying forms.

$$\lim_{n\to\infty} \frac{n}{an^2 - b} = 0$$

Proof. We want to show that we can choose n large enough so that $\left|\frac{n}{an^2 - b}\right| < \varepsilon$. If $n > \sqrt{b/a}$ then the denominator is positive and we can drop the absolute values below. So

$$\begin{aligned} \left|\frac{n}{an^2 - b}\right| = \frac{n}{an^2 - b} &< \varepsilon \\ \implies \qquad \frac{1}{\varepsilon} &< \frac{an^2 - b}{n} \\ &= an - \frac{b}{n} \end{aligned}$$

Rearranging we see that we must choose N large enough so that $n > N$ implies

$$n > \frac{1}{a}\left(\frac{1}{\varepsilon} + \frac{b}{n}\right)$$

But what about the n on the right-hand side? Observe that $b \geq b/n$ for all $n \in \mathbb{N}$. We are now in position to complete the proof. Let $\varepsilon > 0$ and let $N = \max\left\{\sqrt{b/a}, \frac{1}{a}\left(\frac{1}{\varepsilon} + b\right)\right\}$. Then $n > N$ implies

$$an > \frac{1}{\varepsilon} + b > \frac{1}{\varepsilon} + \frac{b}{n}$$

Rearranging we see that

$$\frac{1}{\varepsilon} < an - \frac{b}{n} = \frac{an^2 - b}{n}$$

Since everything is positive we can recipricate to obtain

$$\varepsilon > \frac{n}{an^2 - b} = \left|\frac{n}{an^2 - b} - 0\right|$$

as desired. $\square$

Remark. This proof is made incredibly annoying because of the "subtraction" in the denominator. Indeed, under the same assumptions above, observe that

$$\frac{n}{an^2 + b} < \frac{n}{an^2} = \frac{1}{an} < \varepsilon$$

So now for a given $\varepsilon > 0$ we can choose $N = \frac{1}{a\varepsilon}$ so that $n > N$ implies $\left|\frac{n}{an^2 + b} - 0\right| < \varepsilon$. In other words,

$$\lim_{n\to\infty} \frac{n}{an^2 + b} = 0$$