Verify the limit below without using Stirling's Formula.

$$\lim_{n\to\infty} \frac{(2n)!}{4^n n! n!} = 0$$

Proof. Let $a_n = (2n)!/(4^n n! n!)$. An easy calculation shows that

$$\rho_n =^{\text{def}} \frac{a_n}{a_{n-1}} = \frac{2n-1}{2n} = 1 - \frac{1}{2n}$$

Thus

$$a_{n-1} > \left(1 - \frac{1}{2n}\right) a_{n-1} = a_n$$

So $\{a_n\}$ is a decreasing sequence of positive numbers, and hence, it converges by the Monotone Convergence Theorem.

By the Binomial Theorem,

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{j=0}^{n} \binom{n}{j} \left(\frac{1}{n}\right)^j \\ &= 1 + \binom{n}{1}\frac{1}{n} + \binom{n}{2}\frac{1}{n^2} + \cdots + \binom{n}{n-1}\frac{1}{n^{n-1}} + \frac{1}{n^n} \\ &= 1 + 1 + \text{positive terms} \\ &> 2\end{aligned}$$

It follows that

$$2^{1/n} < 1 + \frac{1}{n} = \frac{n+1}{n}$$

or

$$2^{-1/n} > \frac{n}{n+1} = 1 - \frac{1}{n+1}$$

Thus

$$\rho_n = 1 - \frac{1}{2n} < 2^{-1/(2n-1)} < 2^{-1/2n}$$

Notice that $a_0 = 1$ and write

$$\begin{aligned}a_n &= \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \cdots \frac{a_2}{a_1} \frac{a_1}{a_0} a_0 \\ &= \rho_n \, \rho_{n-1} \cdots \rho_2 \, \rho_1 \\ &< 2^{-1/(2n)} \, 2^{-1/(2n-2)} \cdots 2^{-1/4} \, 2^{-1/2} \\ &= 2^{\frac{-1}{2} \sum_{j=1}^{n} \frac{1}{j}}\end{aligned}$$

Since the Harmonic Series diverges to positive infinity, we see that the last expression approaches 0 as $n \to \infty$. Hence the result follows by the Squeeze Law.

□

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