Throughout this exam you may assume that $A \subseteq \mathbb{R}$ is never the empty set.

1. (10 points) Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers. Suppose that $\sum_{n=1}^{\infty} a_n$ converges and $\limsup b_n < \infty$. Prove that $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

2. (15 points) Use an ε-δ argument to prove that $f(x) = x^2 + 3x$ is continuous at 4.

3. (15 points)

   (a) *Carefully* state the **Intermediate Value Theorem**.

   (b) Let $f : [0,1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$.
   Hint: Consider the function $g(x) = f(x) - f(x + 1/2)$.

4. (10 points) Let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose that $\{x_n\} \subset A$ is a convergent sequence. Prove that $\{f(x_n)\}$ is a bounded sequence.

   Warning: You can not assume that $\lim_{n\to\infty} x_n = c \in A$ since A is not necessarily closed.

5. (15 points) *Carefully* state the **Axiom of Completeness**, and use it to prove that every bounded increasing sequence of real numbers has a limit.

6. (10 points) Find the radius of convergence and give the exact interval of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n$$

7. (10 points) Let $\{a_n\}$ be a sequence of positive numbers. Define

$$\sigma_n = \frac{1}{n}\sum_{j=1}^{n} a_j = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

If $\{a_n\}$ is increasing prove that $\{\sigma_n\}$ is increasing. That is, show that

$$a_n \le a_{n+1} \implies \sigma_n \le \sigma_{n+1} \qquad (3)$$

8. (15 points) Let f be a continuous function on $[0, \infty)$ and suppose that f is uniformly continuous on $[1, \infty)$. Prove that f is uniformly continuous on $[0, \infty)$.

9. (**Bonus -** 10 points) Let $I = [a, b]$ be a closed bounded interval and let $f, g : I \to \mathbb{R}$ be continuous functions. Prove that $C = \{x \in I : f(x) = g(x)\}$ is a closed set.

   Hint: First show that $C_0 = \{x \in I : f(x) = 0\}$ is closed.