Throughout this exam you may assume that $A \subseteq \mathbb{R}$ is never the empty set.

1. (10 points) Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers. Suppose that $\sum_{n=1}^{\infty} a_n$ converges and $\limsup b_n < \infty$. Prove that $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

   Solution:

   We use the Comparison test.

   We claim that $\{b_n\}$ is bounded. Otherwise, there is a subsequence $\{b_{n_k}\}$ such that $b_{n_k} \to \infty$ as $k \to \infty$. Thus $\limsup b_n = \infty$, contrary to our assumption.

   So there is an $M > 0$ such that for all $n \in \mathbb{N}$ we have $0 < b_n < M$. Now $a_n > 0$ implies

   (1) $$0 < a_n b_n < M a_n, \quad n \in \mathbb{N}$$

   By the Algebraic Limit theorems,

   (2) $$\sum_{n=1}^{\infty} M a_n = M \sum_{n=1}^{\infty} a_n < \infty$$

   The result now follows by combining (1) and (2) and invoking the Comparison test.

2. (15 points) Use an ε-δ argument to prove that $f(x) = x^2 + 3x$ is continuous at 4.

   Solution:

   Probably the easiest method is to rewrite (using Taylor polynomials, perhaps)

   $$\begin{aligned} f(x) - 28 &= x^2 + 3x - 28 \\ &= (x-4)^2 + 11(x-4) \end{aligned}$$

   and proceed in the usual manner.

   Here's the standard approach. Let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/12\}$. Then $|x - 4| < \delta$ implies $|x + 7| < 12$ and

   $$\begin{aligned} |f(x) - f(4)| &= |x^2 + 3x - 28| \\ &= |x-4|\,|x+7| \\ &< \frac{\varepsilon}{12}\, 12 \end{aligned}$$

   as desired.

3. (15 points)

(a) *Carefully* state the **Intermediate Value Theorem**.

Solution:

Intermediate Value Theorem Let f be a continuous function on an interval I and let $a, b \in I$ with $a < b$. Then f attains every value between $f(a)$ and $f(b)$. In other words, if $f(a) < L < f(b)$ (or $f(b) < L < f(a)$), then there is $c \in (a, b)$ such that $f(c) = L$.

(b) Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $f(0) = f(1)$. Prove that there exists a point $c \in [0, 1/2]$ such that $f(c) = f(c + 1/2)$.
Hint: Consider the function $g(x) = f(x) - f(x + 1/2)$.

Solution:

Let $g(x) = f(x) - f(x + 1/2)$ and notice that g is continuous on $[0, 1/2]$. Also,

$$g(0) = f(0) - f(1/2)$$

and

$$\begin{aligned} g(1/2) &= f(1/2) - f(1) = f(1/2) - f(0) \\ &= -g(0) \end{aligned}$$

If $f(0) = f(1/2)$ then choose $c = 1/2$. Otherwise, $g(0)$ and $g(1/2)$ have opposite signs. So by the IVT, there is a point $c \in (0, 1/2)$ such that

$$0 = g(c) = f(c) - f(c + 1/2)$$

as desired.

4. (10 points) Let $f : A \to \mathbb{R}$ be uniformly continuous. Suppose that $\{x_n\} \subset A$ is a convergent sequence. Prove that $\{f(x_n)\}$ is a bounded sequence.
Warning: You can not assume that $\lim_{n\to\infty} x_n = c \in A$ since A is not necessarily closed.

Solution:

Let $\varepsilon > 0$ and choose $\delta > 0$ so that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon$. Since $\{x_n\}$ converges, it is a Cauchy sequence. So there exists $N \in \mathbb{N}$ such that $m, n \geq N$ implies $|x_n - x_m| < \delta$. It follows that for all $m, n \geq N$

$$|f(x_n) - f(x_m)| < \varepsilon$$

and hence, $\{f(x_n)\}$ is a Cauchy sequence. But Cauchy sequences are convergent and hence bounded.

5. (15 points) *Carefully* state the **Axiom of Completeness**, and use it to prove that every bounded increasing sequence of real numbers has a limit.

Solution:

Axiom of Completeness: Every nonempty subset of real numbers that is bounded above has a least upper bound.

Proof. Now suppose that $A = \{x_n\}$ is a bounded increasing sequence of real numbers. By the AoC, A has a supremum. So let $x^* = \sup A$. We claim that $\lim_{n\to\infty} x_n = x^*$. To see this let $\varepsilon > 0$. Then, by the definition of supremum, there is an $N \in \mathbb{N}$ such that $x^* - \varepsilon < x_N < x^*$. But since $\{x_n\}$ is an increasing sequence, we must have

$$x^* - \varepsilon \leq x_N < x_n < x^*$$

provided $n \geq N$. It follows that

$$\begin{aligned} & x^* - \varepsilon < x_n < x^* < x^* + \varepsilon \\ \Longrightarrow & -\varepsilon < x_n - x^* < \varepsilon \\ \Longrightarrow & |x_n - x^*| < \varepsilon \end{aligned}$$

as desired. □

6. (10 points) Find the radius of convergence and give the exact interval of convergence for the power series below.

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} x^n$$

Solution:

We use the (Absolute) Ratio Test (the Root Test also works). Let a_n equal the summand. Then

$$\begin{aligned} \rho = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| &= \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^2} \frac{n^2}{3^n} \frac{|x|^{n+1}}{|x|^n} \\ &= 3|x| \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 3|x| \end{aligned}$$

It follows that the Radius of Convergence is $1/3$ and hence the Interval of Convergence is $(-1/3, 1/3)$. Now we test the end points. When $x = 1/3$, the series becomes

$$\sum_{n=1}^{\infty} \frac{3^n}{n^2} \left(\frac{1}{3}\right)^n = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$$

The series also converges at $x = -1/3$. Thus $I = [-1/3, 1/3]$.

7. (10 points) Let $\{a_n\}$ be a sequence of positive numbers. Define

$$\sigma_n = \frac{1}{n}\sum_{j=1}^{n} a_j = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$$

If $\{a_n\}$ is increasing prove that $\{\sigma_n\}$ is increasing. That is, show that

$$a_n \le a_{n+1} \implies \sigma_n \le \sigma_{n+1} \tag{3}$$

Remark. Here's a useful observation. We'll use a baseball analogy. For each $n \in \mathbb{N}$ let a_n denote a players batting average for the n^{th} game in a season. Then σ_n would denote her average through the first n games. Thus (3) is simply stating that if a player's game averages continue to increase, then so does her seasonal average.

Solution:

We postpone the induction proof since there is a much easier way. Notice that

$$a_{n+1} - a_k \ge 0, \quad \text{for } 1 \le k \le n \tag{4}$$

$\{a_n\}$ is increasing. Thus

$$\begin{aligned}
\sigma_{n+1} - \sigma_n &= \frac{1}{n+1}(a_1 + a_2 + \cdots + a_n + a_{n+1}) - \frac{1}{n}(a_1 + a_2 + \cdots + a_n) \\
&= \frac{n(a_1 + a_2 + \cdots + a_n + a_{n+1}) - (n+1)(a_1 + a_2 + \cdots + a_n)}{n(n+1)} \\
&= \frac{na_{n+1} - (a_1 + a_2 + \cdots + a_n)}{n(n+1)} \\
&= \frac{(a_{n+1} - a_1) + (a_{n+1} - a_2) + \cdots + (a_{n+1} - a_n)}{n(n+1)} \ge 0
\end{aligned}$$

The result follows.

Here's an easier proof. First note that

$$\sigma_n = \frac{a_1 + a_2 + \cdots + a_n}{n} \le \frac{n\,a_n}{n} = a_n \le a_{n+1}$$

Thus

$$\begin{aligned}
(n+1)\sigma_n &= n\sigma_n + \sigma_n \\
&\le n\sigma_n + a_{n+1} \\
&= a_1 + a_2 + \cdots + a_n + a_{n+1}
\end{aligned}$$

Now divide through by $n+1$ to obtain the result.

Now here's the induction proof. WARNING: THE INDUCTION HYPOTHESIS IS EMPLOYED IN A WAY THAT IS NOT OBVIOUS WITHOUT SOME EXPERIENCE. I'll begin by using some concrete values for n.

Clearly,

(5)
$$\sigma_1 = a_1 = \frac{a_1 + a_1}{2} \leq \frac{a_1 + a_2}{2} = \sigma_2$$

Now show that $\sigma_2 \leq \sigma_3$ implies $\sigma_3 \leq \sigma_4$. So

$$a_1 + a_2 + a_3 = a_1 + a_2 + a_3$$

and each one of the following inequalities follows directly from the induction hypothesis.

$$a_1 + a_2 + a_4 \geq \frac{3}{2}(a_1 + a_2)$$

$$a_1 + a_3 + a_4 \geq \frac{3}{2}(a_1 + a_3)$$

$$a_2 + a_3 + a_4 \geq \frac{3}{2}(a_2 + a_3)$$

Now add up the left-hand and right-hand sides of the 4 lines above to obtain

$$3a_1 + 3a_2 + 3a_3 + 3a_4 \geq 4a_1 + 4a_2 + 4a_3$$

Thus

$$\sigma_4 = \frac{a_1 + a_2 + a_3 + a_4}{4} \geq \frac{a_1 + a_2 + a_3}{3} = \sigma_3$$

Now the path is clear. We have already proven the base case above (see (5)). So let

$$\Sigma^k = \sum_{j=1}^{n+2} a_j - a_k, \quad 1 \leq k \leq n+1$$

$$\sigma^k = \Sigma^k - a_{n+2}, \quad 1 \leq k \leq n+1$$

That is, Σ^k is the sum of the first $n+2$ terms in the sequence excluding the k^{th} term. For example,

$$\Sigma^3 = a_1 + a_2 + a_4 + \cdots + a_{n+2}$$

and

$$\sigma^3 = a_1 + a_2 + a_4 + \cdots + a_{n+1}$$

Then

$$\Sigma^1 \geq \frac{n+1}{n}\,\sigma^1$$

$$\Sigma^2 \geq \frac{n+1}{n}\,\sigma^2$$

$$\vdots$$

$$\Sigma^{n+1} \geq \frac{n+1}{n}\,\sigma^{n+1}$$

As we did before, add up the left and right sides to obtain.

$$n(a_1 + a_2 + \cdots + a_n) + (n+1)a_{n+2} \geq (n+1)(a_1 + a_2 + \cdots + a_{n+1})$$

Now adding $a_1 + a_2 + \cdots + a_{n+1}$ to both sides yields

$$(n+1)(a_1 + a_2 + \cdots + a_{n+1} + a_{n+2}) \geq (n+2)(a_1 + a_2 + \cdots + a_{n+1})$$

Rearranging we obtain

$$\sigma_{n+2} = \frac{a_1 + a_2 + \cdots + a_{n+2}}{n+2} \geq \frac{a_1 + a_2 + \cdots + a_{n+1}}{n+1} = \sigma_{n+1}$$

...and now my head hurts.

8. (15 points) Let f be a continuous function on $[0, \infty)$ and suppose that f is uniformly continuous on $[1, \infty)$. Prove that f is uniformly continuous on $[0, \infty)$.

Solution:

Notice that by Theorem 3.19.2 (from the text), f is uniformly continuous on the closed and bounded interval $[0, 1]$. Now let $\varepsilon > 0$. By the uniform continuity of f on $[0, 1]$, there exists a $\delta_1 > 0$ such that $x, y \in [0, 1]$ with $|x - y| < \delta_1$ implies $|f(x) - f(y)| < \varepsilon/2$. Also, by the uniform continuity of f on $[1, \infty)$, there exists a $\delta_2 > 0$ such that $x, y \in [1, \infty)$ with $|x - y| < \delta_2$ implies $|f(x) - f(y)| < \varepsilon/2$.

Now let $\delta = \min\{\delta_1, \delta_2\}$. The cases $x, y \in [0, 1]$ or $x, y \in [1, \infty)$ with $|x - y| < \delta$ have already been dealt with above. Now suppose that $x \in [0, 1]$ and $y \in [1, \infty)$ with $|x - y| < \delta$. Then $x \leq 1 \leq y$ and $|x - 1| < \delta_1$ and $|y - 1| < \delta_2$. Thus

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f(1) + f(1) - f(y)| \\ &\leq |f(x) - f(1)| + |f(1) - f(y)| \\ &< \varepsilon/2 + \varepsilon/2 \end{aligned}$$

9. (**Bonus** - 10 points) Let $I = [a, b]$ be a closed bounded interval and let $f, g : I \to \mathbb{R}$ be continuous functions. Prove that $C = \{x \in I : f(x) = g(x)\}$ is a closed set.

Hint: First show that $C_0 = \{x \in I : f(x) = 0\}$ is closed.

Solution:

First we show that $C_0 = \{x \in I : f(x) = 0\}$ is closed. Let c be a limit point of $C_0 \subseteq I$ and let $\{x_n\}$ be a sequence in C_0 that converges to c. Clearly $c \in I$ since I is closed. Now

$$f(c) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0$$

and hence $c \in C_0$. It follows that C_0 is closed.

For the general case, observe that $h(x) = f(x) - g(x)$ is a continuous function. Now apply the above result to the set

$$\{x \in I : h(x) = 0\} = C$$