You may assume that $A \subseteq \mathbb{R}$ is never the empty set.

1. Let A be a bounded subset of real numbers and let $c > 0$. Define $cA = \{ca : a \in A\}$. Show that $\sup cA = c \sup A$.

Solution:

By the Axiom of Completeness, $\alpha = \sup A$ is finite. So for all $a \in A$,

$$\alpha \geq a$$

Then $c > 0$ implies that

$$c\alpha \geq ca \tag{1}$$

It follows that cA is bounded above (by $c\alpha$), hence $\sup cA$ is finite (again by the Axiom of Completeness). Now suppose that β is any upper bound for cA. Then $\beta \geq ca$ for all $a \in A$. In particular, $\beta/c \geq a$ for all $a \in A$. It follows that β/c is an upper bound for A and so, $\beta/c \geq \alpha$ since α is the least upper bound (of A). Hence

$$\beta \geq c\alpha \tag{2}$$

Putting (1) and (2) together, we conclude that $c\alpha = \sup cA$.

2. Let $0 \leq b < 1$. Show that for each $k \in \mathbb{N}$, we have

$$\lim_{n\to\infty} n^k b^n = 0 \tag{3}$$

Solution:

Throughout, we fix $k \in \mathbb{N}$ and let $a_n = n^k b^n$.

Claim: It suffices to show that $a_{n+1}/a_n \to L < 1$ as $n \to \infty$. But this is trivial. Now by the Ratio Test (everything is positive), the series $\sum n^k b^n < \infty$. It follows that $n^k b^n \to 0$ as $n \to \infty$.

Solution (cont):

Some might argue that the above proof is the consequence of a trick. Here's a second proof that makes the above "ratio test" more plausible but does *not* make use of series. Let $R = \frac{L+1}{2}$ and notice that $L < R < 1$. It follows that for n sufficiently large, say $n \geq N \in \mathbb{N}$, we have

$$\frac{a_{N+1}}{a_N} < R \quad \text{or} \quad a_{N+1} < Ra_N$$

It follows that

$$a_{N+2} < Ra_{N+1} < R^2 a_N$$

Continuing we see that

$$0 < a_{N+k} < R^k a_N$$

Now observe that a_N is a constant and hence, the right-hand side of the last inequality approaches 0 as $k \to \infty$. The result now follows by the Squeeze Law. (Cf. this argument with the standard proof of the Ratio Test, which can be found in any calculus text.)

Here's another proof. According to Theorem 2.9.10 (from the text), (3) is equivalent to

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$

Notice that $b^{-1} > 1$ and we may write $b^{-1} = 1 + c$ for some $c > 0$. Then

$$\frac{1}{a_n} = \frac{b^{-n}}{n^k} = \frac{1}{n^k}(1+c)^n$$

Now choose $n > k$ and apply the Binomial Formula (as we have seen before). This yields

$$\begin{aligned}
\frac{1}{a_n} &= \frac{1}{n^k}(1+c)^n \\
&= \frac{1}{n^k}\left(1 + nc + \binom{n}{2}c^2 + \binom{n}{3}c^3 + \cdots + \binom{n}{k+1}c^{k+1} + \text{positive terms}\right) \\
&> \frac{1}{n^k}\binom{n}{k+1}c^{k+1} = \frac{1}{n^k}\, \underbrace{n(n-1)\cdots(n-k+1)}_{k \text{ factors}}(n-k)\, c^{k+1} \\
&= \frac{n(n-1)\cdots(n-k+1)}{n^k}\,(n-k)c^{k+1}
\end{aligned}$$

Letting n go to infinity we see that the rational quantity approaches 1 and hence, the right-hand side grows without bound. In other words, $1/a_n \to \infty$ as $n \to \infty$. We conclude that $\lim_{n\to\infty} a_n = 0$.

3. Decide whether the set below is open, closed, or neither. Provide a *brief* justification. In particular, if the set is neither open nor closed find a limit point that does not belong to the set **and** find a point c in the set such that no ε-neighborhood of c is in the set.

$$\{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\} \cup \{\pi^2/6\}$$

Note: $\sum_{n=1}^{\infty} 1/n^2 = \pi^2/6$

Solution:

Let $A = \{1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N}\}$. Then A contains an strictly increasing (and hence isolated) sequence of real numbers whose only limit point is $\pi^2/6$. It follows that the $A \cup \{\pi^2/6\}$ is closed.

4. Let $\{a_n\}$ and $\{b_n\}$ be sequences of positive real numbers. Suppose that $\lim_{n\to\infty} n^2 a_n = A < \infty$ and $\lim_{n\to\infty} b_n = b$. Prove that $\sum_{n=1}^{\infty} a_n\, b_n$ converges.

Solution:

We first remark that by the Limit Comparison test, $\sum_{n=1}^{\infty} a_n$ converges since $\sum_{n=1}^{\infty} n^{-2}$ converges. Now since $\{b_n\}$ converges, it is bounded. So there is an $L > 0$ such that for all $n \in \mathbb{N}$ we have $0 < b_n < L$. Now $a_n > 0$ implies

$$0 < a_n b_n < L a_n, \quad n \in \mathbb{N} \tag{4}$$

By the Algebraic Limit theorems,

$$\sum_{n=1}^{\infty} L a_n = L \sum_{n=1}^{\infty} a_n < \infty \tag{5}$$

The result now follows by combining (4) and (5) and invoking the Comparison test.

As an alternative, notice that since everything is positive, we can invoke the Limit Comparison test directly.

$$\lim_{n\to\infty} \frac{a_n b_n}{1/n^2} = \lim_{n\to\infty} n^2 a_n \cdot \lim_{n\to\infty} b_n = Ab < 0$$

It follows by the LCT that $\sum a_n b_n < \infty$ since $\sum 1/n^2 < \infty$.

5. Use an ε-δ argument to prove that $f(x) = 2x^3 - 5x$ is differentiable at 2.

Solution:

We claim that f is differentiable at 2 and that $f'(2) = 19$. To see this, let $\varepsilon > 0$ and let $\delta = \min\{1, \varepsilon/14\}$. Then $|x - 2| < \delta$ implies $|x + 4| < 7$ and

$$\begin{aligned}\left|\frac{f(x) - f(2)}{x - 2} - 19\right| &= |2x^2 + 4x + 3 - 19| \\ &= 2|x - 2|\,|x + 4| \\ &< \left(\frac{2}{1}\right)\left(\frac{\varepsilon}{14}\right)\left(\frac{7}{1}\right) = \varepsilon\end{aligned}$$

as desired.

6. Show the series $\sum_{n=0}^{\infty} \frac{x^n}{2-x^n}$ converges for $x \in [0, 1)$. Show that the series converges uniformly on $[0, a]$ for all $0 < a < 1$.

Solution:

Done in class.

7. Let $f(x) = x^2$ if $x \in \mathbb{Q}$ and let $f(x) = 0$ otherwise. Prove f is discontinuous for $x \neq 0$. Prove f is differentiable at 0.

Solution:

We proved continuity in class. Differentiability is nearly as easy. To see the f is discontinuous away from 0, choose $c \neq 0$ and work two cases:

(i) If $c \notin \mathbb{Q}$, then choose a sequence $\{q_n\}$ of rational numbers that convege to c. Now observe that $f(q_n) = q_n^2 \to c^2 \neq 0 = f(c)$.

(ii) On the other hand, if $c \in \mathbb{Q}$, then choose a sequence of irrational numbers $\{i_n\}$ such that $i_n \to c$, and notice that $f(i_n) = 0 \to 0 \neq c^2 = f(c)$.

8. Let $f : [0, 1) \to \mathbb{R}$ be uniformly continuous. Prove that $\lim_{x \to 1^-} f(x)$ exists.
Hint: Use the fact that 1 is limit point of $[0, 1)$.

Solution:

So let $\{x_n\} \subset [0, 1)$ be a sequence such that $x_n \to 1$ as $n \to \infty$. It follows that $\{x_n\}$ is a Cauchy sequence, and hence, so is $\{f(x_n)\}$, as we've saw on Exam 2. Since Cauchy sequences are convergent, the result follows.

9. Let $c_k > 0$ for $k = 1, 2, 3, \dots, n$. Prove that

$$n^2 \le \underbrace{(c_1 + c_2 + \cdots + c_n)}_{A} \underbrace{\left(\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n}\right)}_{B} \tag{6}$$

for all $n \in \mathbb{N}$.

Solution:

Trivially, $1^2 \le c_1 \cdot \dfrac{1}{c_1}$ and as we saw in class,

$$2^2 \le (c_1 + c_2)\left(\frac{1}{c_1} + \frac{1}{c_2}\right)$$

follows immediately from the observation that $(c_1 - c_2)^2 \ge 0$.

Now let A and B be defined as in (6). Then $AB \ge n^2$ and

$$\begin{aligned}
(c_1 + c_2 + \cdots + c_n + c_{n+1})\left(\frac{1}{c_1} + \frac{1}{c_2} + \cdots + \frac{1}{c_n} + \frac{1}{c_{n+1}}\right) &= (A + c_{n+1})\left(B + \frac{1}{c_{n+1}}\right) \\
&= AB + \frac{A}{c_{n+1}} + Bc_{n+1} + 1 \\
&\ge n^2 + 1 + \frac{A}{c_{n+1}} + Bc_{n+1} \qquad (7)
\end{aligned}$$

It suffices to show that $\dfrac{A}{c_{n+1}} + Bc_{n+1} \ge 2n$. We have

$$\begin{aligned}
\frac{A}{c_{n+1}} + Bc_{n+1} &= \frac{c_1}{c_{n+1}} + \frac{c_2}{c_{n+1}} + \cdots + \frac{c_n}{c_{n+1}} + \cdots \\
&\quad + \cdots \frac{c_{n+1}}{c_1} + \frac{c_{n+1}}{c_2} + \cdots + \frac{c_{n+1}}{c_n} \\
&= \left(\frac{c_1}{c_{n+1}} + \frac{c_{n+1}}{c_1}\right) + \left(\frac{c_2}{c_{n+1}} + \frac{c_{n+1}}{c_2}\right) + \cdots + \left(\frac{c_n}{c_{n+1}} + \frac{c_{n+1}}{c_n}\right) \\
&\ge \underbrace{2 + 2 + \cdots + 2}_{n \text{ terms}} = 2n
\end{aligned}$$

Continuing from (7), we have

$$\begin{aligned}
(A + c_{n+1})\left(B + \frac{1}{c_{n+1}}\right) &\ge n^2 + 1 + \frac{A}{c_{n+1}} + Bc_{n+1} \\
&\ge n^2 + 1 + 2n = (n+1)^2
\end{aligned}$$

as desired.

10. Let $\{a_n\}$ be a sequence of nonnegative real numbers. Suppose that $a_n \to a \in \mathbb{R}$ as $n \to \infty$. Prove that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} a_n = a$$

Hint: One may assume that $a = 0$.

Solution:

Let $\varepsilon > 0$. Since $a_n \to 0$ as $n \to \infty$, we can choose N so large that $|a_n| = a_n < \varepsilon$ whenever $n \geq N$. Thus

$$\begin{aligned}
\frac{1}{N+m}\sum_{n=1}^{N+m} a_n &= \frac{1}{N+m}\underbrace{\sum_{n=1}^{N} a_n}_{S_N} + \frac{1}{N+m}\sum_{n=N+1}^{N+m} a_n \\
&\leq \frac{1}{N+m}S_N + \frac{1}{N+m}\sum_{n=N+1}^{N+m}\varepsilon \\
&= \frac{1}{N+m}S_N + \frac{m}{N+m}\varepsilon \\
&< \frac{1}{N+m}S_N + \varepsilon
\end{aligned}$$

Notice that S_N is constant (i.e., it does not depend on m). Now let $m \to \infty$ to conclude that

$$0 \leq \lim_{m\to\infty}\frac{1}{N+m}\sum_{n=1}^{N+m} a_n \leq \varepsilon$$

Since $\varepsilon > 0$ was arbitrary, we conclude that

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n} a_n = 0 \tag{8}$$

For the general case, let $b_n = |a - a_n|$. Then $b_n \geq 0$ for each $n \in \mathbb{N}$ and $b_n \to 0$ as $n \to \infty$. Now

$$\begin{aligned}
\left|\frac{1}{n}\sum_{j=1}^{n} a_n - a\right| &= \frac{1}{n}\left|\sum_{j=1}^{n}(a_n - a)\right| \\
&\leq \frac{1}{n}\sum_{j=1}^{n}|a_n - a| \\
&= \frac{1}{n}\sum_{j=1}^{n} b_n
\end{aligned}$$

Now by (8) the right-hand side goes to zero as $n \to \infty$. The result follows.