It was mentioned after class last week that with the sequential definition of continuity (Definition 1 from the text), it was easier to prove that a function is discontinuous at a point. I claim that this it is still possible, even without Definition 1.

First, let's examine the ε-δ (bare-bones) definition of continuity. Let f be a function defined on a set D. Below we assume that x is always an element of D. We say f is continuous at $c \in D$ provided that

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ such that if } |x - c| < \delta, \text{ then } |f(x) - f(c)| < \varepsilon$$

What is the negation of the above definition? That is, what does it mean to prove that f is not continuous at a point $c \in D$?

$$\exists \varepsilon > 0 \text{ such that } \forall \delta > 0 \; \exists x \text{ satisfying } |x - c| < \delta, \text{with } |f(x) - f(c)| \geq \varepsilon \tag{1}$$

Keep in mind we need only find one domain element for each $\delta > 0$. However, since δ is arbitrary, we must be able to find domain elements, x, that are arbitrarily close c with the property that $|f(x) - f(c)| \geq \varepsilon$.

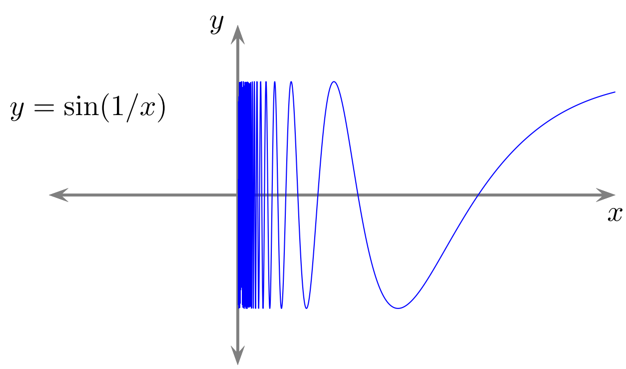

Figure 1: The Topologist's Sine Curve

To illustrate, let us prove that $f(x) = \sin(1/x)$ has no limit at $x = 0$ (Figure 1).

Now let $x_n = \dfrac{2}{(2n+1)\pi}$. Notice that $x_n \to 0$ as $n \to \infty$, and $y_n \stackrel{\text{def}}{=} f(x_n) = -1$ if n is odd and $+1$ if n is even. In particular, the sequence $\{y_n\}$ is not Cauchy and hence can not converge (as we saw in chapter 7). We claim that this is enough to imply (1).

Now let $\varepsilon = 1/2$ (we only need to find one positive ε so that (1) holds). Now we must show that for every choice of $\delta > 0$, we can find an element x with $|x - 0| < \delta$, **but** $|f(x) - L| \geq 1/2$. Wait a minute! What the heck is L? It turns out that it doesn't matter. For the time being let's just assume that L is any real number.

So let $\delta > 0$ **be arbitrary**. By the Archimedian Property there exists an $n \in \mathbb{N}$ such that $2n > \frac{1}{\delta}$ and since $\frac{(4n+1)\pi}{2} > n$, we must have $0 < x_{2n+1} < x_{2n} < \delta$. Now as we noted above, $f(x_n) = (-1)^n$. Now it follows from an argument that we have seen before that at least one of the inequalities below must be true. That is, either

$$|L - f(x_{2n})| = |L - 1| \geq 1/2 \quad \text{or} \quad |L - f(x_{2n+1})| = |L + 1| \geq 1/2$$

as desired!

Let's recap this approach. We showed that if we can find a sequence $\{w_n\}$ that converges to c with the property that the sequence $\{f(w_n)\}$ either does not converge to $f(c)$ (if known), or simply does not converge (to any real number), then f is not continuous at c.

Remark. The above problem is sometimes formulated as follows:

Let

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ L, & \text{otherwise.} \end{cases}$$

where L is usually specified (0 is a popular choice). We used the above approach to show that, for this example, no choice of L will work!