Let $\{s_n\}$ be a sequence of nonnegative numbers, and for each $n \in \mathbb{N}$ let $\sigma_n = \frac{1}{n}(a_1 + a_2 + \cdots + a_n)$. Show that

$$\liminf a_n \le \liminf \sigma_n \le \limsup \sigma_n \le \limsup a_n \tag{1}$$

Remark. We note that by the Squeeze Law, (1) immediately implies that if $\lim_{n\to\infty} a_n = a$, then $\{\sigma_n\}$ converges and $\lim_{n\to\infty} \sigma_n = a$.

Proof

We first establish the last inequality. If the sequence $\{a_n\}$ is not bounded above, then $\limsup a_n = \infty$ and there is nothing to prove.

Let $\alpha_m = \sup\{a_n : n > m\}$. We recall same facts about the supremum and the limit supremum.

(i) $\limsup \sigma_n$ and $\limsup a_n$ both exist as (perhaps) extended real numbers.

(ii) $a_n \le \alpha_N$ for all $n > N$.

Following the hint (from the text), let $n > M > N$. Then

$$\begin{aligned}
\sigma_n &= \frac{1}{n}\sum_{j=1}^{N} a_j + \frac{1}{n}\sum_{j=N+1}^{n} a_j \\
&\le \frac{1}{n}\sum_{j=1}^{N} a_j + \frac{1}{n}\sum_{j=N+1}^{n} \alpha_N \\
&\le \frac{1}{M}\sum_{j=1}^{N} a_j + \frac{n-N}{n}\alpha_N \\
&< \frac{1}{M}\sum_{j=1}^{N} a_j + \alpha_N
\end{aligned}$$

Notice that the right-hand side is now independent of n. In fact, the right-hand side is an upper bound of the set $\{\sigma_{M+1}, \sigma_{M+2}, \ldots\}$. It follows that

$$\sup_{n>M} \sigma_n \le \frac{1}{M}\sum_{j=1}^{N} a_j + \alpha_N$$

Now let $M \to \infty$ to conclude that

$$\limsup \sigma_n \le 0 + \alpha_N$$

Letting $N \to \infty$ yields

$$\limsup \sigma_n \le \limsup a_n \tag{2}$$ ∎

The problem with the above proof is that the technique can not be readily adapted to the first inequality. Also, we can not directly use exercise 2.11.8 because this proof depended heavily on the fact that everything is nonnegative.

We need to try another approach. Before proceeding, let's establish the following result.

Proposition. Suppose that $a_n \le b_n$ for all $n \in \mathbb{N}$. Then

$$\liminf a_n \le \liminf b_n \text{ and } \limsup a_n \le \limsup b_n. \tag{3}$$

Proof
So $b_n - a_n \ge 0$ for all n. In other words, 0 is a lower bound for the set $\{b_1 - a_1, b_2 - a_2, \ldots\}$. It follows that for each $n \in \mathbb{N}$

$$0 \le \inf_{j \ge n}(b_j - a_j)$$

Letting $n \to \infty$ yields

$$0 \le \liminf(b_n - a_n) \tag{4}$$

Now by exercise 2.12.5, we have

$$\begin{aligned}\liminf b_n &= \liminf(b_n - a_n + a_n)\\ &\ge \liminf(b_n - a_n) + \liminf a_n\\ &\ge \liminf a_n\end{aligned}$$

And the last line follows since $\liminf(b_n - a_n) \ge 0$. From (4) we have

$$\begin{aligned}0 &\ge -\liminf(b_n - a_n)\\ &= \limsup(a_n - b_n)\end{aligned}$$

See exercise 2.11.8. Finally,

$$\begin{aligned}\limsup a_n &\le \limsup(a_n - b_n) + \limsup b_n\\ &\le \limsup b_n\end{aligned}$$

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Corollary. If $a_n \le b_n$ for all $n > N \in \mathbb{N}$, then $\liminf a_n \le \liminf b_n$ and $\limsup a_n \le \limsup b_n$.

Now let $\{a_n\}$ be a sequence of real numbers (note the change from above). Then

$$\liminf a_n \le \liminf \sigma_n$$

As before, if $\liminf a_n = -\infty$ or if $\liminf \sigma_n = \infty$, there is nothing to prove.

Let $n > N$ and let $\beta_N = \inf\{a_j : j > N\}$. Then

$$\begin{aligned}
\sigma_n &= \frac{1}{n}\sum_{j=1}^{N} a_j + \frac{1}{n}\sum_{j=N+1}^{n} a_j \\
&\geq \frac{1}{n}\sum_{j=1}^{N} a_j + \frac{1}{n}\sum_{j=N+1}^{n} \beta_N \\
&= \frac{1}{n}\sum_{j=1}^{N} a_j + \frac{n-N}{n}\beta_N \\
&=^{\text{def}} b_n
\end{aligned}$$

Now $\sigma_n \geq b_n$ for all $n > N$. So by (3),

$$\begin{aligned}
\liminf \sigma_n \geq \liminf b_n &= \liminf \left(\frac{1}{n}\sum_{j=1}^{N} a_j + \frac{n-N}{n}\beta_N\right) \\
&\geq \liminf \left(\frac{1}{n}\sum_{j=1}^{N} a_j\right) + \liminf \left(\frac{n-N}{n}\beta_N\right) \\
&= \lim_{n\to\infty} \left(\frac{1}{n}\sum_{j=1}^{N} a_j\right) + \lim_{n\to\infty} \left(\frac{n-N}{n}\beta_N\right) \\
&= \sum_{j=1}^{N} a_j \lim_{n\to\infty} \frac{1}{n} + \beta_N \lim_{n\to\infty} \frac{n-N}{n} \\
&= 0 + \beta_N
\end{aligned}$$

Now recall that β_N is an increasing sequence, and as such, must converge. So we take the limit of both sides to conclude that

$$\liminf \sigma_n \geq \lim_{N\to\infty} \beta_N = \liminf a_n \tag{5}$$

and the proof is complete.

Now that we have proven (5) *without* relying on positivity, it turns out to be rather straight-forward to prove the right-hand side of (1). In fact, the proof is almost trivial.

Once again we take advantage of exercise 2.11.8. By (5) we have

$$\liminf(-\sigma_n) \geq \liminf(-a_n)$$

Rearranging yields

$$\limsup \sigma_n = -\liminf(-\sigma_n) \leq -\liminf(-a_n) = \limsup a_n$$

as desired.

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