The following exercise is a slightly modified version of exercise 3.17.14 from the text.

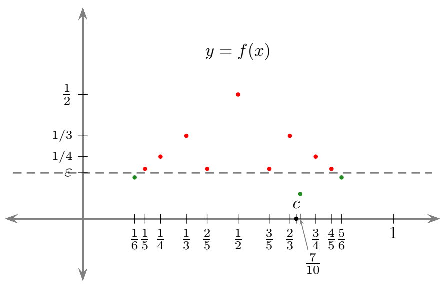
Let $f : (0,1) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1/q, & \text{if } x = p/q \text{ with } p,q \in \mathbb{N},\ p < q, \text{ and } (p,q) = 1; \\ 0, & \text{if } x \in (0,1) \setminus \mathbb{Q}. \end{cases}$$

Here $(p,q) = 1$ means that p and q are relatively prime, i.e., they have no common factors. Throughout this handout, p and q are assumed to be relatively prime.

Show that f is continuous on the irrationals and discontinuous at each of the rationals.

We sketch $y = f(x)$ below.



We claim that f is continuous at $c \in (0,1) \setminus \mathbb{Q}$. To see this fix c and let $\varepsilon > 0$ (e.g., as shown in the sketch).

By the Archimedean Property, there $q_0 \in \mathbb{N}$ such that $\dfrac{1}{q_0} < \varepsilon$. (For the given ε in the sketch, $q_0 = 6$ appears to work.) Now let

(1) $$\delta = \frac{1}{2} \min\{|c - p/q| : p, q \in \mathbb{N},\ p < q < q_0, \text{ and } (p,q) = 1\}$$

Then if $|x - c| < \delta$ there are only two possibilities. If $x \in (0,1) \setminus \mathbb{Q}$ then

$$|f(x) - f(c)| = |0 - 0| < \varepsilon.$$

Otherwise, $x = p/q$ with $q \geq q_0$. In that case,

$$|f(x) - f(c)| = |f(p/q) - 0| = 1/q \leq 1/q_0 < \varepsilon$$

For example, in the sketch above, formula (1) would generate

$$\begin{aligned} \delta &= \frac{1}{2} \min\{|c - p/q| : p, q \in \mathbb{N},\ p < q < 6, \text{ and } (p,q) = 1\} \\ &= \frac{1}{2} \min\{|c - 1/2|, |c - 1/3|, \ldots, |c - 3/5|, |c - 4/5|\} \\ &= \frac{|c - 2/3|}{2} \end{aligned}$$

since, for all the rational numbers that matter, it appears that c is closest to $2/3$.

So, for example, if it turns out that $7/10 \in (c - \delta, c + \delta)$, that's okay since

$$|f(7/10) - f(c)| = 1/10 - 0 < 1/6 < \varepsilon.$$

In other words, the only rational numbers that we care about correspond to the red dots in the sketch, and by the construction of δ, we made sure that none of these are in the interval $(c - \delta, c + \delta)$.

On the other hand, suppose that $c = \frac{p}{q} \in (0,1) \cap \mathbb{Q}$. Now let $1/q > \varepsilon > 0$, then for any $\delta > 0$ we can find an irrational number $r \in (c - \delta, c + \delta)$. Thus

$$|f(c) - f(r)| = f(p/q) - 0 = \frac{1}{q} > \varepsilon$$