Let f be a continuous function on $I = [a, b]$. Show that the function f^* defined by $f^*(x) = \sup\{f(y) : a \leq y \leq x\}$, for all $x \in [a, b]$, is an increasing continuous function on $[a, b]$. See Figure 1.

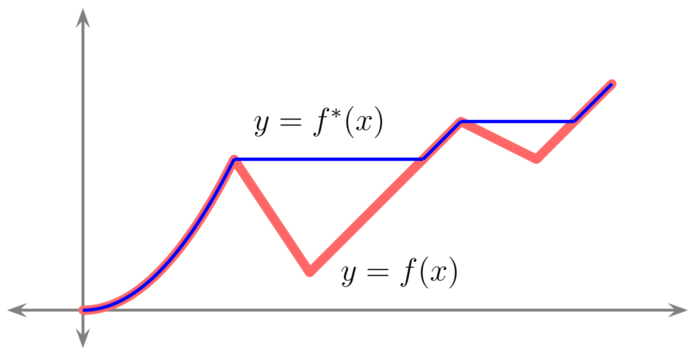

Figure 1: Graph of f versus f^*

First observe that if $A \leq B \leq C \leq D$, then, for example,

(1) $$D - A \geq C - B \geq 0, \text{ etc.}$$

We also note that since f is continuous and $[a, x]$ closed and bounded for any $x \in [a, b]$, that the supremum is actually a maximum (by the Max-Min Theorem). In other words, for each $x \in [a, b]$, $\exists x_0 \in [a, x]$ such that

(2) $$f^*(x) = f(x_0)$$

Finally, note that $f^*(x) \geq f(x)$ for all $x \in [a, b]$.

Let $a \leq s < t \leq b$. Then

$$\{f(y) : a \leq y \leq s\} \subset \{f(y) : a \leq y \leq t\}$$

So by exercise 1.4.7 from the text

$$f^*(s) = \sup\{f(y) : a \leq y \leq s\} \leq \sup\{f(y) : a \leq y \leq t\} = f^*(t)$$

In other words, f^* is increasing.

Method 1 - Show f^* is (Pointwise) Continuous:

Now let $\varepsilon > 0$ and fix $c \in [a, b]$. Then by the continuity of f (at c), there is a $\delta = \delta(c) > 0$ such that

$$|x - c| < \delta \implies |f(x) - f(c)| < \varepsilon/2$$

Case 1: $c < x < c + \delta$. In this case, $f^*(x) \geq f^*(c)$ since f^* is increasing. If we have equality, there is nothing to show. Otherwise, we suppose that $f^*(x) > f^*(c)$. So by (2), $f^*(x) = f(x_0)$ for some $a \leq x_0 \leq x$. However, we must have $x \geq x_0 > c$ else $f^*(x) = f^*(c)$. Thus, $|x_0 - c| < \delta$. Now

$$f(x_0) = f^*(x) > f^*(c) \geq f(c)$$

So by (1),

$$|f^*(x) - f^*(c)| \le |f(x_0) - f(c)| < \varepsilon/2 < \varepsilon$$

Case 2: $c - \delta < x < c$. In this case, $f^*(x) \le f^*(c)$ since f^* is increasing. If we have equality, there is nothing to show. Otherwise, we suppose that $f^*(x) < f^*(c)$. So by (2), $f^*(c) = f(c_0)$ for some $a \le c_0 \le c$. However, we must have $x < c_0 \le c$ else $f^*(x) = f^*(c)$. In other words, we may assume that $|x - c_0| < \delta$. Now

$$f(x) \le f^*(x) < f^*(c) = f(c_0)$$

So by (1),

$$|f^*(x) - f^*(c)| \le |f(x) - f(c_0)| \tag{3}$$

And now we are stuck. Can you explain why? Think about this before proceeding.

The problem is that our earlier choice of $\delta > 0$ specifically depended on c. We don't (immediately) know anything about the $\delta = \delta(c_0) > 0$ required to control the last expression above. Although we can circumvent this by appealing to the uniform continuity of f (since I is closed and bounded), there is an easier way. Notice that since $c - \delta < x < c_0 \le c$, we also have $|c_0 - c| < \delta$. Now continuing with (3),

$$\begin{aligned} |f^*(x) - f^*(c)| &\le |f(x) - f(c_0)| \\ &\le |f(x) - f(c)| + |f(c) - f(c_0)| \\ &< \varepsilon/2 + \varepsilon/2 \end{aligned}$$

It follows that f^* is continuous at c.

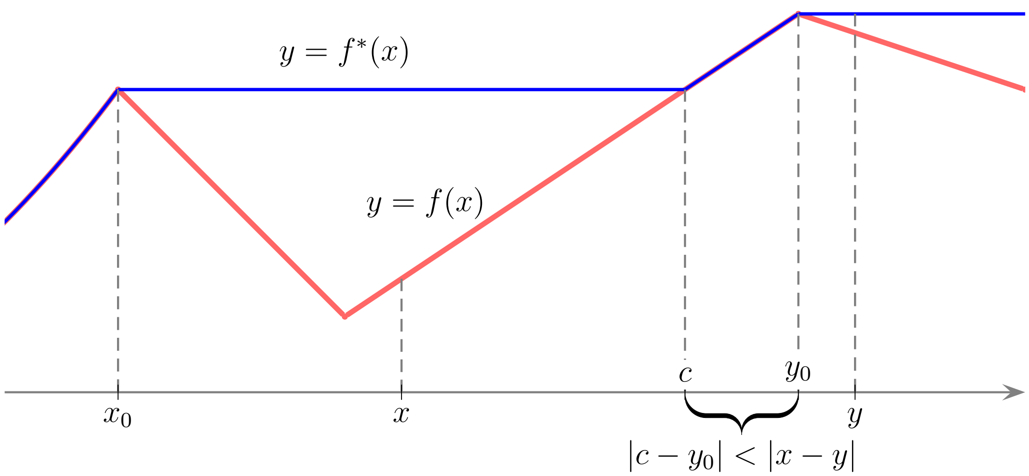

Figure 2: f versus f^*

Method 2: Prove f^* is Uniformly Continuous:

By Theorem 8 (in class), f is uniformly continuous. So let $\varepsilon > 0$. There exists a $\delta > 0$ so that $|y - x| < \delta$ implies $|f(y) - f(x)| < \varepsilon$.

Now let $y > x$ with $|y - x| < \delta$. If $f^*(y) = f^*(x)$ there is nothing to prove. Suppose then that $f^*(y) > f^*(x)$. So there exists y_0 with $x < y_0 \le y$ such that $f^*(y) = f(y_0)$. (Why must $y_0 > x$)? Also, there exists $x_0 \le x$ such that $f^*(x) = f(x_0)$. We have

$$x_0 \le x < y_0 \le y$$

If $x = x_0$ we are done since we now have $|x_0 - y_0| \le |x - y| < \delta$. So by the uniform continuity of f

$$|f^*(x) - f^*(y)| = |f(x_0) - f(y_0)| < \varepsilon$$

On the other hand, suppose that $x_0 < x < y_0 \le y$. Then $f(x) < f(x_0) < f(y_0)$. By the IVP there exists $c \in (x, y_0)$ such that $f(c) = f(x_0)$ (see Figure 2). It follows that

$$|f^*(x) - f^*(y)| = |f(c) - f(y_0)| < \varepsilon$$

since $|c - y_0| < |x - y| < \delta$. Now since uniform continuity implies continuity, we are done.

Remark. Method 2 is easier to follow and it illustrates a very nice use of the IVP.