2.14 Infinite Series

Series and Partial Sums

What does it mean to add up an infinite number of things?

Definition. Infinite Series

An **infinite series** is the sum of an infinite sequence of numbers. Formally, it is

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n$$

For the remainder of this chapter whenever we use the term *series* it should be understood that we are referring to an *infinite series*.

Remark. Warning: Proceed with care when you see the word *formally* in mathematics. Loosely speaking it means “we are writing an expression that may or may not make any sense!”. For example, regardless of any subsequent definitions, the following series does not exist as a real or extended real number as we shall see later.

(1) $$1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} = \sum_{n=0}^{\infty} (-1)^n$$

Definition. Infinite Series, nth Term, Partial Sum, etc.

Given the **infinite series**

(2) $$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

we define the following. The number a_n is called the **nth term** of the series. It is also called the **summand**. The **nth partial sum** of the series is denoted by s_n and is defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ s_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ s_n &= a_1 + a_2 + a_3 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

Notice that the partial sums generate a new sequence, the so-called **sequence of partial sums**, $\{s_n\}$. Now if this new sequence converges to a limit, say $L \in \mathbb{R}$, we say that the series (2) converges and that its **sum** is L. Specifically,

(3) $$s_n \to L \text{ as } n \to \infty \quad \Longrightarrow \quad \sum_{n=1}^{\infty} a_n = L$$

In other words,

(4) $$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n$$

whenever the limit exists. Otherwise, the series **diverges**.

Note: For convenience we occasionally drop the indices. In such cases, $\sum a_n$ is understood to mean $\sum_{n=1}^{\infty} a_n$ whether or not the series converges.

Example 1. Does the series below converge or diverge.

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$$

We claim that the series converges. Using partial fractions, we first rewrite the summand as $\frac{1}{j(j-1)}$. Thus

$$\begin{aligned} s_n &= \sum_{j=2}^{n} \frac{1}{j(j-1)} = \sum_{j=2}^{n} \left(\frac{1}{j-1} - \frac{1}{j} \right) \\ &= \left(\frac{1}{1} - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n} \right) \\ &= 1 - \left(\frac{1}{2} - \frac{1}{2} \right) + \left(\frac{1}{3} - \frac{1}{3} \right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n-1} \right) + \frac{1}{n} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

It follows that the series converges. In fact,

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(1 - \frac{1}{n} \right) = 1$$

Remark. In this example we took advantage of something called a telescoping sum. In general, a **telescoping sum** is a series of the form

$$\begin{aligned} \sum_{j=1}^{n} (a_j - a_{j+1}) &= (a_1 - a_2) + (a_2 - a_3) + (a_3 - a_4) + \cdots + (a_n - a_{n+1}) \\ &= a_1 + (a_2 - a_2) + (a_3 - a_3) + \cdots + (a_n - a_n) - a_{n+1} \\ &= a_1 - a_{n+1} \end{aligned}$$

Now suppose that the sequence $\{a_n\}$ is convergent. That is, suppose that $a_n \to a$ as $n \to \infty$. Then

$$\sum_{n=1}^{\infty} (a_n - a_{n+1}) = \lim_{n\to\infty} \underbrace{(a_1 - a_{n+1})}_{s_n} = a_1 - a$$

Geometric Series

A **geometric series** is a series of the form

(5) $$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1} = \sum_{n=0}^{\infty} ar^n$$

where a and r are fixed constants with $a \neq 0$. The constant r is usually called the **common ratio**.

We wish to obtain a closed formula for (5). Suppose that the series in (5) converges to a real number, call it s. Then

$$s = \sum_{n=0}^{\infty} ar^n = a + \sum_{n=0}^{\infty} ar^{n+1}$$

(6) $$= a + r \sum_{n=0}^{\infty} ar^n = a + rs$$

Thus

(7) $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

Now the right-hand side of (7) is defined for all $r \neq 1$. On the other hand, it is easy to see that the left-hand side of (7) diverges for $|r| > 1$ (Why?). It appears that a bit more care is needed.

Instead, we consider the nth partial sum of $\sum_{k=0}^{\infty} r^k$.

$$\begin{aligned} s_n &= 1 + r + r^2 + \cdots + r^n \\ \implies rs_n &= \quad r + r^2 + r^3 + \cdots + r^{n+1} \end{aligned}$$

Now subtract the second row from the first to obtain

$$s_n - rs_n = 1 - r^{n+1} \quad \text{or}$$

$$s_n = \frac{1 - r^{n+1}}{1-r}$$

Now suppose that $|r| < 1$. Then, by the Common Limits Theorem, $r^{n+1} \to 0$ as $n \to \infty$ and

(8) $$1 + r + r^2 + \cdots + r^n + \cdots \text{ converges to } \frac{1}{1-r}$$

In general, we have

(9) $$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \quad |r| < 1.$$

If $|r| \geq 1$ then the series diverges.

Example 2. Find the following (infinite) sum...if it exists.

$$\sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n$$

Notice that the common ratio is $1/3$. From (9) we conclude that

$$\sum_{n=0}^{\infty} 5\left(\frac{1}{3}\right)^n = \frac{5}{1-1/3}$$

Example 3. Express $2.\overline{325}$ as a ratio of two integers.

The Divergence Test

Notice that whenever $\sum a_n$ converges the terms a_n must approach 0. To see this, let $\{s_n\}$ be the partial sums of the infinite series $\sum a_n$. That is, let

$$s_n = \sum_{k=0}^{n} a_n$$

and suppose that

$$\sum_{n=0}^{\infty} a_n = L, \quad L \in \mathbb{R}$$

Then

$$\lim_{n\to\infty} s_n = L$$

Notice that $a_n = s_n - s_{n-1}$. It follows that

$$\begin{aligned}\lim_{n\to\infty} a_n &= \lim_{n\to\infty} (s_n - s_{n-1}) \\ &= \lim_{n\to\infty} s_n - \lim_{n\to\infty} s_{n-1} \\ &= L - L \\ &= 0\end{aligned}$$

We have

Theorem 1. If $\sum a_n$ converges then $a_n \to 0$ as $n \to \infty$.

Remark. The converse is not true. That is, there are infinite series whose terms go to zero but the series fails to converge. Consider the example below.

Example 4. The Harmonic Series Diverges

That is

(10) $$\sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

To see this, notice that

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{15} + \frac{1}{16} + \cdots$$
$$= 1 + \frac{1}{2} + \underbrace{\frac{1}{3} + \frac{1}{4}}_{2 \text{ terms}} + \underbrace{\frac{1}{5} + \cdots + \frac{1}{8}}_{4 \text{ terms}} + \underbrace{\frac{1}{9} + \cdots + \frac{1}{16}}_{8 \text{ terms}} + \cdots$$
$$> \frac{3}{2} + \underbrace{\frac{1}{4} + \frac{1}{4}}_{2 \text{ terms}} + \underbrace{\frac{1}{8} + \cdots + \frac{1}{8}}_{4 \text{ terms}} + \underbrace{\frac{1}{16} + \cdots + \frac{1}{16}}_{8 \text{ terms}} + \cdots$$
$$= \frac{3}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

In other words, the sequence of partial sums is increasing without bound and (10) is established.

Here's shorter proof. It is easy to show that if $x > 1$, one has

(11) $$\frac{1}{x-1} + \frac{1}{x} + \frac{1}{x+1} > \frac{3}{x}$$

Exercise: Verify this.

No suppose that the harmonic series converged, say to some real number s. Then

$$s = \sum_{n=1}^{\infty} \frac{1}{n}$$
$$= 1 + \left(\frac{1}{3-1} + \frac{1}{3} + \frac{1}{3+1}\right) + \left(\frac{1}{6-1} + \frac{1}{6} + \frac{1}{6+1}\right) + \cdots$$
$$> 1 + 3\left(\frac{1}{3} + \frac{1}{6} + \frac{1}{9} + \cdots\right) = 1 + \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right)$$
$$= 1 + \sum_{n=1}^{\infty} \frac{1}{n}$$
$$= 1 + s$$

This is absurd. We conclude that the harmonic series must diverge.

In the next section we will give a another proof that the harmonic series diverges.

The nth-Term Test for Divergence (the Divergence Test)

If $\lim_{n\to\infty} a_n \neq 0$ then the series $\sum_{n=0}^{\infty} a_n$ diverges.

Note: This is the *contrapositive* of Theorem 1.

For example, the series $\sum_{n=1}^{\infty} \frac{n}{2n+1}$ diverges since

$$\lim_{n\to\infty} \frac{n}{2n+1} = 1/2$$

What does the nth-Term Test for Divergence say about the series

$$\sum_{n=1}^{\infty} \frac{|\sin n|}{n}$$

Nothing! Since $\frac{|\sin n|}{n} \to 0$ as $n \to \infty$, the test does not apply.

Do not underestimate the usefulness of the Divergence Test (and of Theorem 1).

Example 5. Find the sum or show that the series diverges.

$$\sum_{n=1}^{\infty} \ln \frac{n}{2n+1}$$

The following theorem is a direct consequence of the limit theorems in section 9.

Theorem 2. Combining Series

If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. Sum-Difference Rule: $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n = A \pm B$
2. Constant Multiple Rule: $\sum c\, a_n = c \sum a_n = cA$ for any real number c.

Example 6. Find the sum.

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{1-2^{n-1}}{4^n} &= \sum_{n=0}^{\infty} \frac{1}{4^n} - \sum_{n=0}^{\infty} \frac{2^{n-1}}{4^n}\\ &= \frac{1}{1-1/4} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{2^n}{4^n}\\ &= \frac{4}{3} - \frac{1}{2}\sum_{n=0}^{\infty} \frac{1}{2^n}\\ &= \frac{4}{3} - \frac{1}{2}\,\frac{1}{1-1/2}\\ &= \frac{1}{3}\end{aligned}$$

Remark. If $\sum a_n = \infty$, i.e., if the series $\sum a_n$ diverges to infinity, then we can still use the constant multiple rule provided we are careful. In particular, we must avoid indeterminate forms such as $0 \times \infty$ or $\infty - \infty$.

For example, if $c \neq 0$ we can apply the constant multiple rule to conclude that $\sum c\, a_n$ diverges whenever $\sum a_n$ does.

For example,

$$\sum_{n=1}^{\infty} \frac{2}{n} = 2 \sum_{n=1}^{\infty} \frac{1}{n} = 2 \times \infty = \infty$$

So the series diverges.

Cesàro Summability - Increasing the No. of Convergent Series?

We begin with a curious example. Suppose that the series in (1) *did* converge to a real number s. Then

$$\begin{aligned} s &= \sum_{n=0}^{\infty} (-1)^n \\ &= 1 - 1 + 1 - 1 + \cdots \\ &= 1 - (1 - 1 + 1 - 1 + \cdots) \\ &= 1 - s \end{aligned}$$

It follows that

$$\sum_{n=0}^{\infty} (-1)^n = 1/2$$

Of course, this is ridiculous since the series diverges by the nth term test.

Nevertheless, observations such the one given above often have merit as we shall see later. We seek a method to increase the number of “convergent” series.

Given a series $\sum a_n$ and its associated sequence of partial sums $s_n = \sum_{j=0}^{n} a_j$. We define a new sequence, the so-called **Cesàro sum** by

(12)
$$\sigma_n = \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) a_j = \underbrace{\frac{s_0 + s_1 + \cdots s_{n-1}}{n}}_{\substack{\text{average of the} \\ \text{1st } n \text{ partial sums}}} = \frac{1}{n} \sum_{j=0}^{n-1} s_j$$

Note: Cesàro sums represents an “averaging” process. In 1890 the Italian mathematician Ernesto Cesàro used such sums while investigating products of infinite series.

Definition. Cesàro Summability

A series $\sum a_n$ is called **Cesàro summable** if its Cesàro sums converge. That is, if

(13)
$$\lim_{n \to \infty} \sigma_n = L \in \mathbb{R}$$

Example 7. Let's compute the Cesàro sums of the divergent series from (1). The even partial sums are $s_{2n} = 1$ and the odd partial sums are $s_{2n+1} = 0$. It follows that

$$\begin{aligned} \sigma_{2n+1} &= \frac{1}{2n+1}(1 + 0 + 1 + 0 + \cdots + 1) \\ &= \frac{n+1}{2n+1} \\ \sigma_{2n} &= \frac{1}{2n}(1 + 0 + 1 + \cdots + 0) = \frac{1}{2} \end{aligned}$$

Hence

$$\lim_{n \to \infty} \sigma_{2n} = \lim_{n \to \infty} \sigma_{2n+1} = 1/2$$

It follows that the divergent series in (1) is *Cesàro summable* to $1/2$.

The next theorem shows that Cesàro summable series converge to the “right” limit whenever the (original) series converges.

Theorem 3. Suppose that $\sum a_n$ is a convergent series with sum, say L. Then $\sum a_n$ is Cesàro summable to L. Specifically, let $s_n = \sum_{j=0}^{n} a_j$. Then

(14)
$$\lim_{n \to \infty} s_n = L \quad \Longrightarrow \quad \lim_{n \to \infty} \sigma_n = L$$

Proof. Let $\varepsilon > 0$. So there is a positive integer N such that $n \geq N$ implies $|s_n - L| < \varepsilon$. Writing $n = N + m$ we have

$$
\begin{aligned}
|\sigma_{N+m} - L| &= \left| \frac{1}{N+m} \sum_{j=0}^{N+m-1} s_j - \frac{N+m}{N+m} L \right| \\
&= \frac{1}{N+m} \left| \sum_{j=0}^{N+m-1} s_j - \sum_{j=0}^{N+m-1} L \right| \\
&\leq \frac{1}{N+m} \sum_{j=0}^{N+m-1} |s_j - L| \\
&= \frac{1}{N+m} \left(\sum_{j=0}^{N-1} |s_j - L| + \sum_{j=N}^{N+m-1} \underbrace{|s_j - L|}_{\text{less than } \varepsilon} \right) \\
&\leq \frac{1}{N+m} \left(\sum_{j=0}^{N-1} |s_j - L| + m\,\varepsilon \right) \\
&< \underbrace{\frac{1}{N+m} \sum_{j=0}^{N-1} |s_j - L|}_{\text{Independent of } m} + \varepsilon \qquad (15)
\end{aligned}
$$

Now let $m \to \infty$ to conclude that

$$0 \leq \limsup |\sigma_{N+m} - L| \leq \varepsilon$$

It follows that

$$\liminf(\sigma_n - L) = \limsup(\sigma_n - L) = 0$$

The result now follows by Theorem 2.10.7.

□

Note: It is also possible to complete the proof of Theorem 3 in the usual way. We continue with the notation from the previous proof. Having chosen N as before, we choose $M \in \mathbb{N}$ so that

$$\frac{1}{N+M} \sum_{j=0}^{N-1} |s_j - L| < \varepsilon$$

Then for all $n \geq N + M$ we have

$$
\begin{aligned}
|\sigma_n - L| &= \frac{1}{n} \left| \sum_{j=0}^{n-1} s_j - \sum_{j=0}^{n-1} L \right| \\
&\leq \frac{1}{n} \sum_{j=0}^{n-1} |s_j - L| \\
&= \frac{1}{n} \left(\sum_{j=0}^{N-1} |s_j - L| + \sum_{j=N}^{n-1} |s_j - L| \right) \\
&\leq \frac{1}{n} \left(\sum_{j=0}^{N-1} |s_j - L| + (n-N)\,\varepsilon \right) \\
&= \frac{1}{n} \sum_{j=0}^{N-1} |s_j - L| + \frac{n-N}{n} \varepsilon \\
&< \frac{1}{N+M} \sum_{j=0}^{N-1} |s_j - L| + \varepsilon \\
&< 2\varepsilon
\end{aligned}
$$

as desired.

To reiterate, the theorem shows that convergent series are necessarily Cesàro summable and the Cesàro sum is equal to the original limit. However, the converse is not true as we saw in Example 7.

We finish with a curious follow-up to Example 7. Recall that under questionable reasoning one might conclude that the divergent series from (1) "converges" to $1/2$. In fact, this was debated in Euler's time (see Guido Grandi's 1703 paper). It was Cesàro and his contemporaries that added rigor to such a conclusion by defining new types of convergence criteria. As we mentioned earlier, these were called summability methods.

As we saw above, we now can say that the *divergent* series $\sum_{n=0}^{\infty}(-1)^n$ is Cesàro summable to $1/2$.

Now consider the product $(1-1+1-1+\cdots)^2$. It is not unreasonable to argue that

$$(1-1+1-1+\cdots)^2 \overset{?}{=}_C (1/2)^2 = 1/4 \tag{16}$$

and to justify such a conclusion using our new summability methods. That is, we should be able to show that $(1-1+1-1+\cdots)^2$ is Cesàro summable to $1/4$. Unfortunately,

$$\begin{aligned}(1-1+1-1+\cdots)^2 &= (1-1+1-1+\cdots)\times(1-1+1-1+\cdots)\\ &= 1-2+3-4+5+\cdots\\ &= \sum_{n=0}^{\infty}(-1)^n n\end{aligned}$$

is not Cesàro summable (to anything). It turns out that the series is *Abel* summable to $1/4$. We will have more to say about this example and other types of summability later.

Example 8. Show that the formula above for $(1-1+1-1+\cdots)^2$ is valid. Also, show that its Cesàro sums σ_n diverge by showing $\sigma_n \to 1/2$ or $-1/2$ depending on the parity of n. We leave this as an exercise.

Properties of Series and Convergence Tests

Theorem 4. Combining Series

If $\sum_n a_n = A$ and $\sum_n b_n = B$ are convergent series, then

i. Sum-Difference Rule: $\sum_n (a_n \pm b_n) = \sum_n a_n \pm \sum_n b_n = A \pm B$

ii. Constant Multiple Rule: $\sum_n c\,a_n = c\sum_n a_n = cA$ for any real number c.

The theorem states, for example, that the sum (or difference) of two convergent series is also convergent.

Theorem 5. Cauchy Criterion for Series. A series converges if and only if, for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $n > m \geq N$ implies

$$\left|\sum_{j=m}^{n} a_j\right| < \varepsilon \tag{17}$$

Proof. Observe that the left-hand side of (17) is

$$\left|\sum_{j=m}^{n} a_j\right| = \left|\sum_{j=1}^{n} a_j - \sum_{j=1}^{m-1} a_j\right| = |s_n - s_{m-1}|$$

So the result is an immediate consequence of the Cauchy Criterion for Sequences.

□

Here is another useful observation about convergent series.

Suppose that the series $\sum a_j$ converges, say $\sum a_j = L$ for some real number L. Then for any $\varepsilon > 0$ there is a positive integer N such that $n \geq N$ implies

$$\begin{aligned}\varepsilon &> |s_n - L|\\ &= \left|\sum_{j=1}^{n} a_j - \sum_{j=1}^{\infty} a_j\right|\\ &= \left|\sum_{j=1}^{n} a_j - \left(\sum_{j=1}^{n} a_j + \sum_{j=n+1}^{\infty} a_j\right)\right|\end{aligned}$$

Rearranging yields

$$\left|\sum_{j=n+1}^{\infty} a_j\right| < \varepsilon$$

In other words, the tail-end of the series can be made arbitrarily small. In fact, we can say more. We have

Theorem 6.

$$\sum_{j=1}^{\infty} a_j \text{ converges} \iff \lim_{m\to\infty} \sum_{j=m}^{\infty} a_j = 0$$

Proof. We have already proven necessity (left to right). Since we don't have a candidate limit in mind, how do we prove sufficiency?

By Theorem 5, it is enough to prove that the series is Cauchy. Now let $\varepsilon > 0$. Then there is a positive integer N such that for all $m \geq N$

$$\left|\sum_{j=m}^{\infty} a_j\right| < \varepsilon/2$$

Now let $n > m \geq N$. Then

$$\begin{aligned}\left|\sum_{j=m}^{n} a_j\right| &= \left|\sum_{j=m}^{n} a_j + \sum_{j=n+1}^{\infty} a_j - \sum_{j=n+1}^{\infty} a_j\right| \\ &\leq \left|\sum_{j=m}^{\infty} a_j\right| + \left|\sum_{j=n+1}^{\infty} a_j\right| \\ &\leq \varepsilon/2 + \varepsilon/2\end{aligned}$$

□

Theorem 7. The Comparison Test

Let $\sum a_n$ be a series with no negative terms.

i. $\sum a_n$ converges if there is a convergent series $\sum c_n$ with $a_n \leq c_n$ for all $n \geq N$ for some positive integer N.
ii. $\sum a_n$ diverges if there is a divergent series $\sum d_n$ with $a_n \geq d_n \geq 0$ for all $n \geq N$ for some positive integer N.

Proof. For (i) notice that

$$\left|\sum_{j=m}^{n} a_j\right| = \sum_{j=m}^{n} a_j \leq \sum_{j=m}^{n} c_j = \left|\sum_{j=m}^{n} c_j\right|$$

Hence the result is an immediate consequence of Theorem 5. The proof of (ii) is left as an exercise.

□

Example 9. Which of the following series can be tested using the Comparison Test? Can you draw any conclusions about the others?

a. $\sum_{n=5}^{\infty} \frac{1}{n+1}$

b. $\sum_{n=5}^{\infty} \frac{1}{n-1}$

c. $\sum_{n=5}^{\infty} \frac{1}{(n+1)^2}$

d. $\sum_{n=5}^{\infty} \frac{1}{(n-1)^2}$

Example 10. One can show

(18) $$\int_0^{\infty} \frac{\sin x}{x}\, dx = \pi/2$$

However, this improper integral is not absolutely convergent. To see this let n be a positive integer. Then for all $x \in [\pi n, \pi(n+1)]$

$$\frac{1}{x} \geq \frac{1}{\pi(n+1)}$$

since $1/x$ is decreasing on $(0, \infty)$. Hence

$$\frac{|\sin x|}{x} \geq \frac{|\sin x|}{\pi(n+1)}$$

It follows that

$$\begin{aligned}
\int_{\pi n}^{\pi(n+1)} \frac{|\sin x|}{x}\, dx &> \frac{1}{\pi(n+1)} \int_{\pi n}^{\pi(n+1)} |\sin x|\, dx \\
&= \frac{1}{\pi(n+1)} |\cos \pi(n+1) - \cos \pi n| \\
&= \frac{2}{\pi(n+1)}
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^{\infty} \frac{|\sin x|}{x}\, dx &= \int_0^{\pi} \frac{|\sin x|}{x}\, dx + \int_{\pi}^{\infty} \frac{|\sin x|}{x}\, dx \\
&> \sum_{n=1}^{\infty} \int_{\pi n}^{\pi(n+1)} \frac{|\sin x|}{x}\, dx \\
&> \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty
\end{aligned}$$

It follows that the integral in (18) is not absolutely convergent.

We are now in position to handle absolute values. A series $\sum a_n$ is said to converge **absolutely** if $\sum |a_n|$ converges.

Theorem 8. The Absolute Convergence Test. If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof. For each $n \in \mathbb{N}$

$$
\begin{aligned}
-|a_n| \le a_n \le |a_n| \\
0 \le a_n + |a_n| \le 2|a_n|
\end{aligned}
$$

Now by Theorem 4, if $\sum |a_n|$ converges then so does $\sum 2|a_n|$. So by the Comparison Test, $\sum (a_n + |a_n|)$ also converges.

Thus

$$
\begin{aligned}
\sum_n a_n &= \sum_n a_n + (|a_n| - |a_n|) \\
&= \sum_n (a_n + |a_n|) - \sum_n |a_n|
\end{aligned}
$$

is the difference of two convergent series, and hence, convergent.

□

Corollary 9. If $\sum_{n=1}^{\infty} |a_n|$ converges then

$$
\left| \sum_{n=1}^{\infty} a_n \right| \le \sum_{n=1}^{\infty} |a_n|
$$

The Integral Test

In the next few sections we consider series without any *negative* terms. In this case, there is only one type of divergence, namely, if the series does not converge it is because the sequence of partial sums increases to infinity.

Nondecreasing Partial Sums

Suppose that $\sum_{n=0}^{\infty} a_n$ is an infinite series of nonnegative terms, that is, $a_n \ge 0$ for $n \ge 1$ then each partial sum is greater than or equal to its predecessor since

$$
\begin{aligned}
s_{n+1} &= s_n + a_{n+1} \\
&\ge s_n \ \text{(since } a_{n+1} \ge 0)
\end{aligned}
$$

It follows that $\{s_n\}$ is a nondecreasing sequence. That is,

$$
s_0 \le s_1 \le s_2 \le \cdots \le s_{n+1} \le \cdots
$$

By the Monotone Convergence Theorem, a nondecreasing sequence converges if and only if it is bounded from above. We have the following

Theorem 10. Convergence of Series with Nonnegative Terms

A series $\sum a_n$ with nonnegative terms converges if and only if its partial sums are bounded from above.

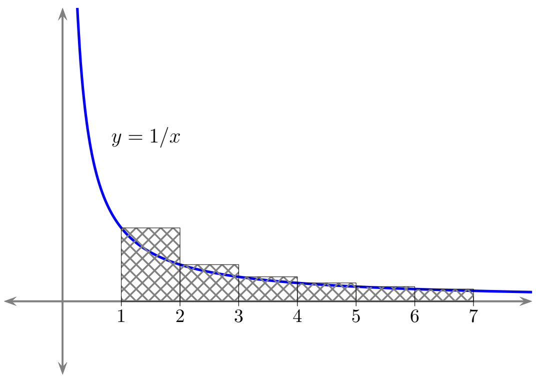
Remark. Because of this theorem and the preceding remarks, it is customary to indicate that a given series **converges** by using the following notation,

$$
\sum a_n < \infty
$$

We will address more general series below.

The Integral Test

Example 11. Does the harmonic series converge? Earlier we proved the divergence of this series using a “condensation” technique. Let’s try to compare the series $\sum_{n=1}^{\infty} 1/n$ to the improper integral $\int_1^{\infty} dx/x$.



From the sketch we see that for each $m \geq 1$

$$\begin{aligned} s_m &= \frac{1}{1} + \frac{1}{2} + \cdots + \frac{1}{m} \\ &\geq \int_1^{m+1} \frac{dx}{x} \end{aligned}$$

It follows that

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \sum_{k=1}^{m} \frac{1}{k} \geq \int_1^{m+1} \frac{dx}{x} \tag{19}$$

holds for all $m \geq 1$. Letting $m \to \infty$ implies that

$$\sum_{n=1}^{\infty} \frac{1}{n} \geq \int_1^{\infty} \frac{dx}{x}$$

But the last quantity is infinite since it is a p-integral (with $p \leq 1$). It follows that the harmonic series diverges.

Remark. The right-hand inequality in (19) plays a prominent role in the computation of Euler’s constant γ, which is defined by the limit

$$\lim_{n\to\infty} \left(\sum_{k=i}^{n} \frac{1}{k} - \ln n \right) = \gamma \approx 0.5772\ldots$$

This is a very important number, but unlike some of the better known constants such as π and e, it is not known whether γ is rational or irrational.

The last example suggests the following theorem.

Theorem 11. The Integral Test

Let $\{a_n\}$ be a sequence of positive terms. Suppose that f is a continuous, positive, decreasing function for all $x \geq N$ (N a positive integer) and that for all $n \geq N$, $a_n = f(n)$. Then the series

$$\sum_{n=N}^{\infty} a_n \text{ and the integral } \int_N^{\infty} f(x)\,dx$$

both converge or both diverge.

Note: The proof depends on the fact that the function is decreasing.

Example 12. Show that the following series diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $f(x) = 1/(x \ln x)$. Then

$$\begin{aligned} f'(x) &= -\frac{1+\ln x}{(x \ln x)^2} \\ &< 0 \end{aligned}$$

for all $x \geq 3$, say. Now

$$\begin{aligned} \int_3^{\infty} \frac{dx}{x \ln x} &= \lim_{B\to\infty} \int_3^{B} \frac{dx}{x \ln x} \\ &= \lim_{B\to\infty} \int_{\ln 3}^{\ln B} \frac{du}{u}, \quad (u = \ln x) \\ &= \int_{\ln 3}^{\infty} \frac{du}{u} \\ &= \infty \end{aligned}$$

The last result follows from the p-integral result established in second semester calculus. It follows that the series diverges by the Integral Test.

p-Series

We are now able to deduce the convergence (or divergence) of a whole class of series, namely, the p-series.

Example 13. p**-Series**

The p**-series**

(20)
$$\sum_{n=1}^{\infty} \frac{1}{n^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

The result is obvious if $p \leq 0$, for then the series diverges by the nth-term test. Notice that the case $p = 1$ yields the Harmonic Series, which we showed was divergent.

If $p > 0$ we can establish the result by appealing to the integral test. In this case, the function $f(x) = 1/x^p$ is clearly continuous and positive on $(0, \infty)$. The first derivative test can be used to show that f is decreasing for $x \geq 1$.

Remark. It is worthwhile to establish a specific case *without* appealing to the Integral Test. We give two (similar) proofs.

In Example 1, we showed that

$$\sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1$$

by discovering a formula for n^{th} partial sum and taking the limit.

Version 1. Now let

$$S_n = \sum_{j=2}^{n} \frac{1}{j(j-1)}$$

and observe that

$$\begin{aligned} S_n &= \sum_{j=2}^{n} \frac{1}{j(j-1)} \\ &< \sum_{j=2}^{n} \frac{1}{j(j-1)} + \frac{1}{n(n+1)} \\ &= S_{n+1} \end{aligned}$$

In particular, for $n > 1$,

$$S_n \leq \sum_{j=2}^{\infty} \frac{1}{j(j-1)} = 1$$

For $n > 1$ we have

$$n(n-1) < n^2 \quad \Longrightarrow \quad \frac{1}{n^2} < \frac{1}{n(n-1)}$$

So that

$$\begin{aligned} \sum_{j=1}^{n} \frac{1}{j^2} &= 1 + \sum_{j=2}^{n} \frac{1}{j^2} \\ &< 1 + \sum_{j=2}^{n} \frac{1}{j(j-1)} \\ &\leq 1 + \sum_{j=2}^{\infty} \frac{1}{j(j-1)} = 2 \end{aligned}$$

Clearly, the left-hand side is an increasing sequence (of partial sums) and it is bounded above. So by the Monotone Convergence Theorem, the series converges.

Version 2. We continue with the notation from above. For $n > m > 2$ we have

$$n(n-1) < n^2 \quad \Longrightarrow \quad \frac{1}{n^2} < \frac{1}{n(n-1)}$$

So that

$$\sum_{j=m}^{n} \frac{1}{j^2} = \sum_{j=m}^{n} \frac{1}{j^2} < \sum_{j=m}^{n} \frac{1}{j(j-1)}$$

Now according to the Cauchy Criterion for Series (Theorem 5), $\{S_n\}$ is a Cauchy Sequence. Hence, so is

$$T_n = \sum_{j=2}^{n} \frac{1}{j^2}$$

and the result follows by once again appealing to Theorem 5.

Remark. It turns out that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

This was first discovered by Euler and is discussed below (see the Basel problem).

The Cauchy Condensation Test

Buried in the original proof of the divergence of the harmonic series is the following useful technique.

Theorem 12. Cauchy Condensation Test

Let $\{a_n\}$ be a nonincreasing sequence of positive terms that converges to 0. Then

(21) $$\sum a_n < \infty \text{ iff } \sum 2^n a_{2^n} < \infty$$

Proof. Since $\{a_n\}$ is decreasing, we have

$$\begin{aligned} &\underbrace{a_2 + a_2} + \underbrace{a_4 + a_4 + a_4 + a_4} + \underbrace{a_8 + a_8 + \cdots + a_8} + \cdots \\ &\geq \underbrace{a_2 + a_3} + \underbrace{a_4 + a_5 + a_6 + a_7} + \underbrace{a_8 + a_9 + \cdots + a_{16}} + \cdots \\ &\geq a_2 + \underbrace{a_4 + a_4} + \underbrace{a_8 + a_8 + a_8 + a_8} + \cdots \end{aligned}$$

Thus

(22) $$\sum_{n=1}^{N} 2^n a_{2^n} \geq \sum_{n=2}^{2^{N+1}} a_n \geq \frac{1}{2} \sum_{n=1}^{N} 2^n a_{2^n}$$

Now suppose that $\sum 2^n a_{2^n} = L < \infty$. Let n be any positive integer and choose N so that $2^{N+1} > n$. Then the left-hand side of (22) implies

$$\begin{aligned}\sum_{k=2}^{n} a_k &< \sum_{k=2}^{2^{N+1}} a_k \\ &\le \sum_{k=1}^{N} 2^k a_{2^k} \\ &\le \sum_{k=1}^{\infty} 2^k a_{2^k} \\ &= L\end{aligned}$$

It follows that for each $n \geq 1$,

$$\begin{aligned}s_n &= a_1 + \sum_{k=2}^{n} a_k \\ &< a_1 + L < \infty\end{aligned}$$

So the nondecreasing sequence of partial sums $\{s_n\}$ is bounded above. Hence by Theorem 10 the series $\sum a_n$ converges. The converse is proven in a similar manner.

Let's redo Example 12.

Example 14. Show that the following series diverges.

(23)
$$\sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

Let $a_n = 1/(n \ln n)$. As we saw in Example 12, $\{a_n\}$ is a decreasing sequence of positive terms. Clearly $1/(n \ln n) \to 0$ as $n \to \infty$.

Now

$$\begin{aligned}\sum 2^n a_{2^n} &= \sum 2^n \frac{1}{2^n \ln 2^n} \\ &= \sum \frac{1}{n \ln 2} \\ &= \frac{1}{\ln 2} \sum \frac{1}{n} \\ &= \infty\end{aligned}$$

So by the Cauchy Condensation Test, the series in (23) must also diverge.

Remark. The above string of equalities is not quite correct. We don't know if we can "factor out" the $\ln 2$ from summand because the series in question *diverges* (to infinity).

So suppose that $\sum_{n=N}^{\infty} a_n = \infty$ and $c > 0$ then

$$\begin{aligned}\sum_{k=N}^{\infty} c\,a_k &= \lim_{n\to\infty} \sum_{k=N}^{n} c\,a_k \\ &=^{?} c \lim_{n\to\infty} \sum_{k=N}^{n} a_k = c \sum_{k=N}^{\infty} a_k \\ &= c \times \infty = \infty\end{aligned}$$

Since the latter series does not converge to a real number, it would appear that we have traded one problem for another. However, it is a simple matter to use the ε-N definition for sequences diverging to infinity to show that if $\lim_{n\to\infty} s_n = \infty$ and $c > 0$ then

$$\lim_{n\to\infty} c\,s_n = c \lim_{n\to\infty} s_n = c \times \infty = \infty$$

And the questionable step above is justified. It follows that $\sum 1/(n \ln 2) = \infty$ and the Example 14 argument is correct.

Example 15. Logarithmic p-Series

What can you say about the following series?

(24) $$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$$

First show that **logarithmic** p-integral

(25) $$\int_2^{\infty} \frac{dx}{x(\ln x)^p}$$

converges if and only if $p > 1$ (cf. example 12).

Now use the integral test to establish that the log p-series converges if and only if $p > 1$. Don't forget to verify that the $f(x) = 1/x(\ln x)^p$ is decreasing on the interval (a, ∞) for some $a \geq 2$.

We mention a useful observation about convergent series.

Suppose that the series $\sum a_n$ converges. That is, suppose

$$\sum_{n=0}^{\infty} a_n = L$$

for some real number L. Then for any $\varepsilon > 0$ there is a positive integer N such that

$$\begin{aligned} \varepsilon &> |s_N - L| \\ &= \left|\sum_{n=0}^{N} a_n - \sum_{n=0}^{\infty} a_n\right| \\ &= \left|\sum_{n=0}^{N} a_n - \left(\sum_{n=0}^{N} a_n + \sum_{n=N+1}^{\infty} a_n\right)\right| \end{aligned}$$

In other words, the tail-end of the series can be made arbitrarily small. That is,

$$\left|\sum_{n=N+1}^{\infty} a_n\right| < \varepsilon$$

We have the following.

Theorem 13.

$$\sum_{n=0}^{\infty} a_n = L \iff \lim_{m\to\infty} \sum_{n=m}^{\infty} a_n = 0$$

Proof. We have already proven necessity (left to right). Since we don't have a candidate limit in mind, how do we prove sufficiency?

Let $s_n = \sum_{k=0}^{n} a_k$. Since $\mathbb{R}$ is complete, it is enough to prove the $\{s_n\}$ is a Cauchy sequence. Now let $\varepsilon > 0$. Then there is an N such that for all $m \geq N$

$$\left|\sum_{k=m}^{\infty} a_k\right| < \varepsilon/2$$

Now let $n > m \geq N$. Then

$$\begin{aligned} |s_n - s_m| &= \left|\sum_{k=0}^{n} a_k - \sum_{j=0}^{m} a_j\right| \\ &= \left|\sum_{k=m+1}^{n} a_k\right| \\ &= \left|\sum_{k=m+1}^{n} a_k + \sum_{k=n+1}^{\infty} a_k - \sum_{k=n+1}^{\infty} a_k\right| \\ &\leq \left|\sum_{k=m+1}^{\infty} a_k\right| + \left|\sum_{k=n+1}^{\infty} a_k\right| \\ &\leq \varepsilon/2 + \varepsilon/2 \end{aligned}$$

□

More Comparison Tests

In this section we extend the Comparison Test.

The Limit Comparison Test

Theorem 14. Limit Comparison Test

Let $a_n > 0$ and $b_n > 0$ for all $n \geq N$ (N an integer).

a. Suppose that $\frac{a_n}{b_n} \to \delta \in [0, \infty)$. If $\sum b_n$ converges then so does $\sum a_n$.

b. Suppose that $\frac{a_n}{b_n} \to \delta \in (0, \infty]$. If $\sum b_n$ diverges then so does $\sum a_n$.

Proof. Suppose first that $a_n/b_n \to \delta \in (0, \infty)$. Let $\varepsilon = \delta/2 > 0$. Then there is a positive integer N such for all $n \geq N$

$$\left|\frac{a_n}{b_n} - \delta\right| < \frac{\delta}{2}$$
$$\implies \frac{\delta}{2} < \frac{a_n}{b_n} < \frac{3\delta}{2}$$
$$\implies \frac{\delta}{2}\, b_n < a_n < \frac{3\delta}{2}\, b_n \tag{26}$$

Now suppose that $\sum b_n < \infty$. Then $\sum 3\delta\, b_n/2 < \infty$ and the right-hand inequality from (26) implies that $\sum a_n < \infty$ by the Comparison Test. On the other hand, if $\sum b_n = \infty$ then the left-hand inequality in (26) implies that $\sum a_n = \infty$, again by the Comparison Test.

The cases when $\delta = 0$ or ∞ are left as exercises.

Corollary 15. Suppose that $a_n > 0$ and $c_n > 0$ with $\lim_{n\to\infty} c_n = L$. Then

(a) If the series $\sum a_n$ converges and $L \in [0, \infty)$ then $\sum a_n\, c_n$ must also converge.

(b) If the series $\sum a_n$ diverges and $L \in (0, \infty]$ then $\sum a_n\, c_n$ must also diverge.

Proof. This is an immediate consequence of the LCT.

□

Example 16. Which of the following series converge? Which diverge? Justify your response.

a. $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+9}}$

b. $\sum_{n=1}^{\infty} \frac{n+1}{n2^n}$

c. $\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$

d. $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$

Earlier we mentioned that $\sum_{n=1}^{\infty} n^{-2} = \pi^2/6$. This important result was discovered by Euler in 1735 as the solution to the so-called Basel problem. In general, the p-series is special case of one of the most important functions in number theory, perhaps in all of mathematics, the Riemann Zeta function. Let $s \in \mathbb{C}$, that is let s be a complex number and denote the real part of s by $\Re(s)$. Then the **Riemann Zeta** function is defined by the infinite series

(27) $$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

The series converges on the (complex) half-plane $\Re(s) > 1$ (and can be defined for all complex numbers except 1).

Euler's result is equivalent to the following proposition.

Proposition 16.

(28) $$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

The shortest proof of (28) requires knowledge of Fourier series. Euler's original proof is easy to find in the library or on the web. We'll try another route; one which exploits the mathematics that have in hand. We need two lemmas.

Lemma 17. *Let n be a positive integer. Then*

(29) $$\int_0^{\pi/2} \cos^{2n} x \, dx = \frac{(2n)!}{4^n n! n!} \frac{\pi}{2}$$

We temporarily postpone the proof of Lemma 17.

Lemma 18. *Let n be a positive integer and let*

(30) $$I_n = \int_0^{\pi/2} \cos^{2n} x \, dx \quad \textit{and} \quad J_n = \int_0^{\pi/2} x^2 \cos^{2n} x \, dx$$

Then

(31) $$I_n = n(2n-1)J_{n-1} - 2n^2 J_n$$

The proof is left as an exercise. (*Hint:* Try integrating I_n by parts, twice.)

Proof. (of Proposition 16) Continuing with the notation from (30). First observe that $J_0 = \pi^3/24$. Also, (29) and (31) imply

$$\frac{(2n)!}{4^n n! n!} \frac{\pi}{2} = n(2n-1)J_{n-1} - 2n^2 J_n$$

Rearranging yields

$$\begin{aligned}
\frac{\pi}{4n^2} &= \frac{2 \cdot 4^{n-1}(n-1)!(n-1)!}{(2n)!} \left(n(2n-1)J_{n-1} - 2n^2 J_n\right) \\
&= \frac{4^{n-1} 2n(2n-1)(n-1)!(n-1)!}{(2n)(2n-1)(2n-2)!} J_{n-1} - \frac{4 \cdot 4^{n-1} n(n-1)! n(n-1)!}{(2n)!} J_n \\
&= \frac{4^{n-1}(n-1)!(n-1)!}{(2n-2)!} J_{n-1} - \frac{4^n n! n!}{(2n)!} J_n \stackrel{\text{def}}{=} K_{n-1} - K_n
\end{aligned}$$

Summing we obtain

$$\begin{aligned}
\frac{\pi}{4} \sum_{n=1}^{N} \frac{1}{n^2} &= \sum_{n=1}^{N} (K_{n-1} - K_n) \\
&= K_0 - K_N \\
&= J_0 - \frac{4^N N! N!}{(2N)!} J_N
\end{aligned}$$

It follows that

(32) $$\sum_{n=1}^{N} \frac{1}{n^2} = \frac{4}{\pi} \frac{\pi^3}{24} - \frac{4}{\pi} \frac{4^N N! N!}{(2N)!} J_N$$

So it is enough to show that the second term on the right-hand side approaches 0 as $N \to \infty$.

Remark. In the next section we will show that

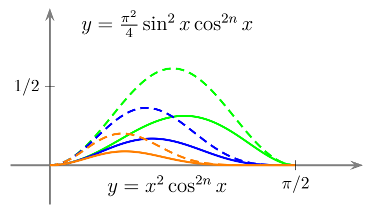
$$a_N = \frac{4^N N! N!}{(2N)!} \sim \sqrt{\pi N} \quad \text{as } N \to \infty$$

That is, we will show that $\lim_{N\to\infty} a_N/\sqrt{\pi N} = 1$. So it would be enough to prove the following little oh result

$$J_n = o(1/\sqrt{n}) \quad \text{as } n \to \infty$$

However, we will opt for the more direct approach shown below.

The sketch shows the graphs of $y = x^2 \cos^{2n} x$ and the "majorizing" functions $y = (\pi^2/4)\sin^2 x \cos^{2n} x$ for $n = 2, 4,$ and 8 (shown in green, blue, and orange, resp.).



Exercise: Let $f(x) = \dfrac{\pi}{2} \sin x - x$. Prove that $f(x) > 0$ for all $0 < x < \pi/2$.

It follows by the exercise that for all $x \in (0, \pi/2)$,

$$\begin{aligned}
x^2 \cos^{2n} x &< \frac{\pi^2}{4} \sin^2 x \cos^{2n} x \\
&= \frac{\pi^2}{4}(1 - \cos^2 x) \cos^{2n} x
\end{aligned}$$

Hence

$$\begin{aligned} J_N &\le \frac{\pi^2}{4}\int_0^{\pi/2}(1-\cos^2 x)\cos^{2N} x\,dx \\ &= \frac{\pi^2}{4}(I_N - I_{N+1}) \\ &= \frac{\pi^3}{8}\left(\frac{(2N)!}{4^N N!N!} - \frac{(2N+2)(2N+1)}{4(N+1)(N+1)}\,\frac{(2N)!}{4^N N!N!}\right) \\ &= \frac{\pi^3}{8}\,I_N\left(1-\frac{2N+1}{2(N+1)}\right) \\ &= \frac{\pi^3}{16}\,\frac{I_N}{N+1} \end{aligned}$$

It follows that

$$0 < \frac{4^N N!N!}{(2N)!}\,J_N \le \frac{\pi^3/16}{N+1}$$

So by the Squeeze Law, the middle term in the last expression approaches 0 as $N \to \infty$. Together with (32) this implies

$$\begin{aligned} \sum_{n=1}^{\infty}\frac{1}{n^2} &= \lim_{N\to\infty}\sum_{n=1}^{N}\frac{1}{n^2} \\ &= \frac{4}{\pi}\,\frac{\pi^3}{24} - \frac{4}{\pi}\,\lim_{N\to\infty}\frac{4^N N!N!}{(2N)!}J_N \\ &= \frac{\pi^2}{6} - 0 \end{aligned}$$

as desired. □

Example 17. Which of the following series converge? Which diverge? Justify your response.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{3^{n-1}+2}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{\sqrt[n]{n}}{n^2}$

c. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{1+2+3+\cdots+n}$

d. $\displaystyle\sum_{n=1}^{\infty}\frac{(\ln n)^2}{n^3}$

Proof. (of Lemma 17) We proceed by induction on n.

For $n = 1$ we leave it as easy exercise to show that

$$\int_0^{\pi/2} \cos^{2(1)} x\,dx = \frac{(2(1))!}{4^1\,1!\,1!}\,\frac{\pi}{2} = \frac{\pi}{4}$$

Now suppose that (29) is true. We need to show that

$$\int_0^{\pi/2} \cos^{2(n+1)} x\,dx = \frac{(2(n+1))!}{4^{n+1}(n+1)!(n+1)!}\,\frac{\pi}{2} \tag{33}$$

As we did in the proof of Proposition 16, let I_n denote the integral in (29). Then the reduction formula for cosine yields

$$\begin{aligned} 2I_{n+1} &= 2\int_0^{\pi/2} \cos^2 x \cos^{2n} x\,dx \\ &= \int_0^{\pi/2} (1+\cos 2x)\cos^{2n} x\,dx \\ &= I_n + \int_0^{\pi/2} \cos 2x \cos^{2n} x\,dx \\ &= I_n + J \end{aligned}$$

For J, integration by parts (with $u = \cos^{2n} x$, $dv = \cos 2x\,dx$, etc.) and the double-angle sine formula gives

$$\begin{aligned} J &= \frac{1}{2}\cos^{2n} x \sin 2x \,\Big|_0^{\pi/2} \\ &\quad + n\int_0^{\pi/2} \sin x \sin 2x \cos^{2n-1} x\,dx \\ &= 0 + 2n\int_0^{\pi/2} \sin^2 x \cos^{2n} x\,dx \\ &= 2n\int_0^{\pi/2} (1-\cos^2 x)\cos^{2n} x\,dx \\ &= 2nI_n - 2nI_{n+1} \end{aligned}$$

Putting this all together leads to

$$2I_{n+1} = I_n + J = (2n+1)I_n - 2nI_{n+1}$$

Rearranging yields

$$(2n+2)I_{n+1} = (2n+1)I_n$$

Thus

$$\begin{aligned} I_{n+1} &= \frac{2n+1}{2(n+1)} I_n = \frac{2(n+1)(2n+1)}{4(n+1)^2} I_n \\ &= \frac{(2n+2)(2n+1)}{4(n+1)^2}\,\frac{(2n)!}{4^n n!n!}\,\frac{\pi}{2} \end{aligned}$$

which is (33).

□

The Ratio and Root Tests

It turns out that there are several other methods that are often used to determine the convergence (or divergence) of an infinite series.

Theorem 19. The Ratio Test

Let $\sum a_n$ be a series of positive terms and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \rho \tag{34}$$

Then

a. the series *converges* if $\rho < 1$,
b. the series *diverges* if $\rho > 1$ or ρ is infinite,
c. the test is *inconclusive* if $\rho = 1$.

Remark. Notice that if the limit in (34) is finite, the series $\sum a_n$ (eventually) behaves like a geometric series with common ratio ρ. We will exploit this idea to prove the theorem.

Part (a) of the theorem immediately yields

Corollary 20. Suppose that $\{a_n\}$ is a sequence of positive numbers such that $a_{n+1}/a_n \to L \in [0,1)$ as $n \to \infty$. Then

$$\lim_{n\to\infty} a_n = 0$$

Proof. Earlier we saw that the terms of a convergent series must approach 0. □

Proof. (of Ratio test) We first prove part a. Suppose $\rho < r < 1$ and let $\varepsilon = r - \rho > 0$. Since

$$\frac{a_{n+1}}{a_n} \to \rho$$

the is a positive integer N such that for all $n \geq N$

$$\left|\frac{a_{n+1}}{a_n} - \rho\right| < \varepsilon = r - \rho$$

hence

$$\frac{a_{n+1}}{a_n} - \rho < r - \rho$$

$$a_{n+1} < r\, a_n, \quad \text{for all } n \geq N.$$

In particular,

$$\begin{aligned} a_{N+1} &< r\, a_N \\ a_{N+2} &< r\, a_{N+1} < r^2\, a_N \\ &\vdots \\ a_{N+m} &< r^m\, a_N \end{aligned}$$

Now let $c_n = r^{N-n}\, a_N$ for all $n \geq N$. Then

$$\sum_{n=N}^{\infty} c_n = \sum_{n=N}^{\infty} a_N\, r^{N-n}$$

is a convergent geometric series since $|r| < 1$. Now since $a_n \leq c_n$ for all $n \geq N$, it follows by the Comparison Test that the series $\sum a_n$ converges.

□

The case when $\rho > 1$ is proven in a similar fashion and is left as an exercise. The example below shows that the case $\rho = 1$ is certainly inconclusive.

Example 18. The Ratio Test is Inconclusive if $\rho = 1$.

Recall that

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n}}_{a_n} \text{ diverges}$$

while

$$\sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2}}_{b_n} \text{ converges.}$$

However,

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{b_{n+1}}{b_n} = 1$$

Remark. Experience (practice) will show that the ratio test is the most useful when applied to series whose terms contain factorials and/or exponentials.

Theorem 21. The Root Test

Let $\sum a_n$ be a series with $a_n \geq 0$ for $n \geq N$ and suppose that

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \rho$$

Then

a. the series *converges* if $\rho < 1$,
b. the series *diverges* if $\rho > 1$ or ρ is infinite,
c. the test is *inconclusive* if $\rho = 1$.

Remark. You can find a more general version of the Ratio and Root Tests in the text.

Example 19. Which of the following series converge? Which diverge? Justify your response.

a. $\sum_{n=1}^{\infty} n^2 e^{-n}$

b. $\sum_{n=1}^{\infty} \frac{n!}{10^n}$

c. $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

d. $\sum_{n=1}^{\infty} \frac{n!}{n^n}$

You may have noticed that many examples in chapter 10 involve factorials. Unfortunately, they are not always easy to manipulate when computing limits unless you happen to be working with quotients of factorials. Why? In fact, what is the factorial function? Is there some easier way to work with it?

Let n be a nonnegative integer. The **factorial** is defined by rule

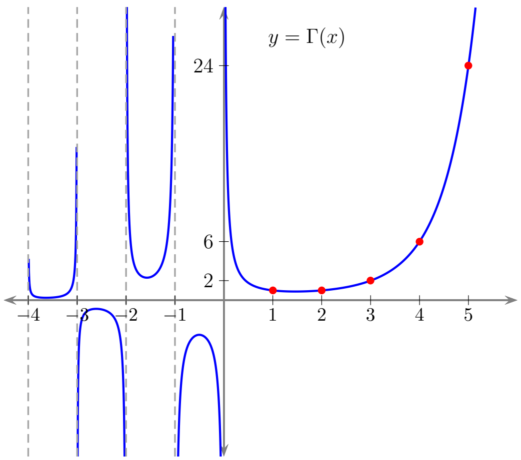
$$0! = 1$$
$$n! = n(n-1)\cdots 2\cdot 1, \quad n > 0$$

At the beginning of the 18th century, mathematicians set about looking to (continuously) extend the factorial to non-integer values. As usual it was Euler who found the solution.

Definition. The Gamma Function

(35) $$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

Here the (improper) integral converges absolutely for all $x \in \mathbb{R}$ except for the non-positive integers. In fact, the Gamma function can be extended throughout the complex plane (again, except for the non-positive integers).



Observe that

$$\Gamma(1) = \int_0^\infty t^0 e^{-t}\, dt$$
$$= \frac{-1}{e^t}\bigg|_0^\infty = 0 - (-1) = 1$$

and for positive integers n, integration by parts yields the recursive relation

$$\begin{aligned}\Gamma(n+1) &= \int_0^\infty t^n e^{-t}\, dt \\ &= -t^n e^{-t}\bigg|_0^\infty + n\int_0^\infty t^{n-1} e^{-t}\, dt \\ &= 0 + n\Gamma(n)\end{aligned}$$

and Euler had found his extension. That is, for each nonnegative integer n, he could now define the factorial by

(36) $$n! = \Gamma(n+1)$$

The gamma function shows up in numerous formulas and important identities. For example, we have the so-called reflection formula

$$\Gamma(x)\,\Gamma(1-x) = \frac{\pi}{\sin \pi x}$$

This leads to the amusing identity

$$\frac{1}{x\Gamma(x)\,\Gamma(1-x)} = \frac{\sin \pi x}{\pi x}$$

which relates the gamma function to the **sinc** function.

About the same time James Stirling discovered an *asymptotic* formula for the factorial function. Although his formula is only an approximation, these estimates do improve as $n \to \infty$ making the formula well suited for estimating the factorial for large integers.

Theorem 22. Stirling's Formula

(37) $$n! = \Gamma(n+1) \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$$

Here the symbol $f(n) \sim g(n)$ means that $\lim_{n\to\infty} f(n)/g(n) = 1$.

Let's use Stirling's formula to verify one of the common limits from section 2.9.

$$\lim_{n\to\infty}\frac{x^n}{n!}=\lim_{n\to\infty}\frac{x^n}{1}\frac{1}{n!}\underbrace{\left(\frac{n}{e}\right)^n\sqrt{2\pi n}\left(\frac{e}{n}\right)^n\frac{1}{\sqrt{2\pi n}}}_{\text{convenient form of } 1}$$

$$=\underbrace{\lim_{n\to\infty}\frac{1}{n!}\left(\frac{n}{e}\right)^n\sqrt{2\pi n}}_{=1 \text{ by Stirling's Formula}}\lim_{n\to\infty}\frac{x^n}{1}\left(\frac{e}{n}\right)^n\frac{1}{\sqrt{2\pi n}}$$

$$=1\cdot\lim_{n\to\infty}\left(\frac{e\,x}{n}\right)^n\frac{1}{\sqrt{2\pi n}}=0\cdot 0$$

Example 20. Evaluate $\lim_{n\to\infty}\frac{4^n n! n!}{(2n)!}$.

First we try it *without* Stirling's Formula. Doh!

On the other hand,

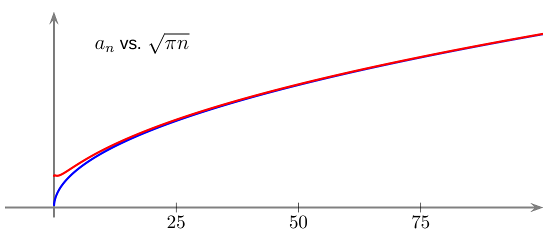
$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}\frac{4^n n! n!}{(2n)!}$$

$$=\frac{2\pi}{\sqrt{2\pi}}\lim_{n\to\infty}\frac{4^n}{1}\left(\frac{n}{e}\right)^n\left(\frac{n}{e}\right)^n\frac{n}{\sqrt{2n}}\left(\frac{e}{2n}\right)^{2n}$$

$$=\sqrt{\pi}\lim_{n\to\infty}\frac{4^n\sqrt{n}}{1}\frac{e^{2n}}{e^n\,e^n}\frac{n^n\,n^n}{n^{2n}}\frac{1}{2^{2n}}$$

$$=\sqrt{\pi}\lim_{n\to\infty}\sqrt{n}=\infty$$

As an added benefit we see that $a_n\sim\sqrt{\pi n}$ as $n\to\infty$.



Example 21. Which of the following series converge? Which diverge? Justify your response.

a. $\displaystyle\sum_{n=1}^{\infty}\frac{n+1}{n!}$

b. $\displaystyle\sum_{n=1}^{\infty}\frac{2^n}{(2n)!}$

c. $\displaystyle\sum_{n=2}^{\infty}\frac{n}{(\ln n)^{(n/2)}}$

d. $\displaystyle\sum_{n=1}^{\infty}\frac{4^n n! n!}{(2n)!}$

e. $\displaystyle\sum_{n=1}^{\infty}\frac{(2n)!}{4^n n! n!}$