3.17 Limits and Continuity

Definition. Limits

Let f be defined on $D \subset \mathbb{R}$. We say that $f(x)$ has limit L as x approaches $c \in \mathbb{R}$, provided that for every $\varepsilon > 0$ there is a $\delta = \delta(c, \varepsilon) > 0$ such that if $x \in D$ and

(1) $$0 < |x - c| < \delta \quad \text{then} \quad |f(x) - L| < \varepsilon$$

In this case we write

(2) $$\lim_{x \to c} f(x) = L$$

Remark.

i. c need not be an element in $D = \mathrm{Dom}(f)$.
ii. Verifying (2) by way of (1) is called an ε-δ argument.
iii. Notice that it's enough to show that (1) holds for some positive ε_1 with $\varepsilon_1 < \varepsilon_1$. For example, we sometimes assume that $1 \geq \varepsilon > 0$.

Example 1. Use an ε-δ argument to prove $\lim_{x \to 4} 3x = 12$.

Proof. Given $\varepsilon > 0$, we must find a $\delta > 0$ so that (1) holds. It is important to observe that δ will usually depend on the given ε and the value of c (4 in this problem). We claim that $\delta = \varepsilon/3$ will suffice. Now suppose that $0 < |x - 4| < \delta = \varepsilon/3$. Then

$$|f(x) - L| = |3x - 12| = 3|x - 4| < 3 \times \frac{\varepsilon}{3} = \varepsilon$$

□

Definition. Continuity

Let f be defined on $D \subset \mathbb{R}$. We say that $f(x)$ is **continuous** at $c \in D$, provided that for every $\varepsilon > 0$ there is a $\delta = \delta(c, \varepsilon) > 0$ such that if $x \in D$ and

(3) $$|x - c| < \delta \quad \text{then} \quad |f(x) - f(c)| < \varepsilon$$

Equivalently, we say f is continuous at c provided that

(4) $$\lim_{x \to c} f(x) = f(c)$$

Compare (3) and (4) with (1) and (2). In particular, notice that it is necessary (but not sufficient) that $c \in D = \mathrm{Dom}(f)$ in the definition of continuity.

Remark.

i. A function is simply called **continuous** if it is continuous at each point in its domain.

ii If f is not continuous at c, we say that f is **discontinuous** at c and that c is a **point of discontinuity** of f.

Example 2. Let $f(x) = 3x$. Use an ε-δ argument to prove that f is continuous at 4.

Since $f(4) = 12$ we actually proved this in Example 1. That is, we proved that

$$\lim_{x \to 4} f(x) = 12 = f(4).$$

The following inequality is used below.

Example 3. Let $0 < \varepsilon < 1$. Prove that $\sqrt{4 + \varepsilon} - 2 < 2 - \sqrt{4 - \varepsilon}$.

Since $0 < \varepsilon < 1$ it follows that

$$0 < 16 - \varepsilon^2 < 16$$

$$\sqrt{16 - \varepsilon^2} < 4$$

$$2\sqrt{16 - \varepsilon^2} < 8$$

$$(4 + \varepsilon) + 2\sqrt{16 - \varepsilon^2} + (4 - \varepsilon) < 16$$

$$\left(\sqrt{4 + \varepsilon} + \sqrt{4 - \varepsilon}\right)^2 < 16$$

$$\sqrt{4 + \varepsilon} + \sqrt{4 - \varepsilon} < 4$$

$$\sqrt{4 + \varepsilon} - 2 < 2 - \sqrt{4 - \varepsilon}$$

Example 4. Prove that $g(x) = x^2$ is continuous at 2.

Proof. Given $1 > \varepsilon > 0$. We let $\delta = \sqrt{4+\varepsilon} - 2$ (which is positive since $\varepsilon > 0$). Now to verify (3), we consider two cases. First suppose that $2 < x < 2 + \delta$. Then

(5) $$0 < x - 2 < \delta = \sqrt{4+\varepsilon} - 2$$

so that

$$\begin{aligned} &2 < x < \sqrt{4+\varepsilon} \\ &4 < x^2 < 4 + \varepsilon \\ &0 < x^2 - 4 < \varepsilon \end{aligned}$$

Now suppose that $2 - \delta < x < 2$. Rearranging yields

(6) $$0 < 2 - x < \delta$$

Now by Example 3, $\delta < 2 - \sqrt{4-\varepsilon}$. Thus

$$0 < 2 - x < 2 - \sqrt{4-\varepsilon} \quad \Longleftrightarrow \quad -2 < -x < -\sqrt{4-\varepsilon}$$

Rearranging yields

$$\begin{aligned} &\sqrt{4-\varepsilon} < x < 2 \\ &4 - \varepsilon < x^2 < 4 \\ &-\varepsilon < x^2 - 4 < 0 \end{aligned}$$

Now (5) and (6) are equivalent to $0 < |x-2| < \delta$. We have shown that

$$0 < |x-2| < \delta \quad \Longrightarrow \quad \left|x^2 - 4\right| < \varepsilon.$$

Since $g(2) = 4$ we have proven that if $|x - 2| < \delta$, then

$$|g(x) - g(2)| = \left|x^2 - 4\right| < \varepsilon$$

as desired. □

Here is an easier argument. Notice that if x is "near" 2, then $x+2$ should be bounded (above and below). More precisely, if $0<\delta\le 1$ say, then $|x-2|<\delta$ implies

$$\begin{aligned} &-\delta < x-2 < \delta \\ \implies & 4-\delta < x+2 < 4+\delta < 5 \\ \implies & |x+2| < 5 \end{aligned}$$

Now let's revisit the previous example. Once again show that $g(x)=x^2$ is continuous at 2.

Proof. Let $\varepsilon>0$. Now choose $\delta=\min\{1,\varepsilon/5\}$. Then if x satisfies $|x-2|<\delta$ we have

$$\begin{aligned} |x^2-4| &= |x+2||x-2| \\ &\le 5|x-2| \qquad \text{(since } \delta\le 1) \\ &< 5\left(\frac{\varepsilon}{5}\right) \qquad \text{(since } \delta\le\varepsilon/5) \\ &= \varepsilon \end{aligned}$$

□

Remark. Notice that the number 5 was unimportant. The important point is that $|x+2|$ is bounded by some $M>0$.

Theorem 1. Let f be a continuous function and k be any real number. Then $k \cdot f$ and $|f|$ are also continuous functions.

The following proposition will be used frequently when dealing with limits of continuous functions.

Proposition 2. Suppose that f is continuous at c and that $f(c) > L$ for some $L \in \mathbb{R}$. Then there is a $\delta > 0$ such that $f(x) > L$ for all $x \in (c - \delta, c + \delta) \cap \mathrm{Dom}(f)$.

Proof. By considering the continuous function $f(x) - L$, it is enough to prove the special case when $L = 0$. Now let $\varepsilon = f(c)/2 > 0$. Since f is continuous at c, there is a $\delta > 0$ such that $|x - c| < \delta$ and $x \in \mathrm{Dom}(f)$ implies

$$-\varepsilon < f(x) - f(c) < \varepsilon$$

Focusing on the left inequality we see that

$$-\frac{f(c)}{2} < f(x) - f(c)$$

In other words

$$f(x) > \frac{f(c)}{2}$$

□

The next result will help us create new continuous functions from old ones.

Theorem 3. Suppose that f and g are continuous at $c \in Dom(f) \cap Dom(g)$. Then so are $f \pm g$, fg, and f/g (provided $g(c) \neq 0$).

Proof. You can find proofs of the first two results in the text. The proof that f/g is continuous will be broken up into 3 parts. The motivation for the (sometimes) mysterious choices for ε below was presented during class.

Part 1. Apply Theorem 2 to the continuous function $|g(x)|$ to conclude there is $\delta_0 > 0$ such that $|g(x)| > |g(c)|/2 > 0$ whenever $|x - c| < \delta_0$. It follows that

(7) $$|x - c| < \delta_0 \quad \Longrightarrow \quad \frac{1}{|g(x)|} < \frac{2}{|g(c)|}$$

Part 2. Now let $\varepsilon > 0$. By the continuity of f, there is a $\delta_1 > 0$ such that

$$|f(x) - f(c)| < \frac{|g(c)|\varepsilon}{4}$$

whenever $|x - c| < \delta_1$. Also, by the continuity of g, there is a $\delta_2 > 0$ such that

$$|g(x) - g(c)| < \frac{|g(c)|^2\varepsilon}{4(|f(c)| + 1)}$$

whenever $|x - c| < \delta_2$.

Part 3. Now let $\delta = \min\{\delta_0, \delta_1, \delta_2\}$. Then $|x - c| < \delta$ implies

$$\begin{aligned}
\left|\frac{f(x)}{g(x)} - \frac{f(c)}{g(c)}\right| &= \left|\frac{f(x)g(c) - f(c)g(x)}{g(x)g(c)}\right| \\
&= \frac{1}{|g(x)g(c)|}|f(x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(x)| \\
&< \frac{2}{|g(c)|^2}|f(x)g(c) - f(c)g(c)| + |f(c)g(c) - f(c)g(x)| \qquad (8) \\
&= \frac{2|g(c)|}{|g(c)|^2}|f(x) - f(c)| + \frac{2|f(c)|}{|g(c)|^2}|g(c) - g(x)| \\
&< \frac{2}{|g(c)|}\,\frac{|g(c)|\varepsilon}{4} + \frac{2|f(c)|}{|g(c)|^2}\,\frac{|g(c)|^2\varepsilon}{4(|f(c)| + 1)} \qquad (9) \\
&= \frac{\varepsilon}{2} + \frac{|f(c)|\varepsilon}{2(|f(c)| + 1)} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \qquad (10)
\end{aligned}$$

Line (8) follows from Part 1 since $\delta < \delta_0$. Line (9) follows from Part 2 since $\delta < \delta_1$ and $\delta < \delta_2$. Finally, line (10) follows since

$$\frac{|f(c)|}{2(|f(c)| + 1)} < 1$$

□