3.18 Properties of Continuous Functions

In section 17, we saw that $|f|$ is continuous whenever f is continuous. In fact, we can say more.

Theorem 1. Suppose that g is continuous at x_0 and that f is continuous at $g(x_0)$, then the composition $f \circ g$ is continuous at x_0. In other words, the composition of continuous functions is a continuous function.

Remark. The hypotheses imply that $\mathrm{Ran}(g) \subset \mathrm{Dom}(f)$.

Proof. Let $\varepsilon > 0$. Also, let $y = g(x)$ and $y_0 = g(x_0)$. By the continuity of f (at y_0), there is a $\delta_f > 0$ such that $y \in \mathrm{Dom}(f)$ and

$$|y - y_0| < \delta_f \quad \Longrightarrow \quad |f(y) - g(y_0)| < \varepsilon.$$

By the continuity of g (at x_0), there is a $\delta_g > 0$ such that $x \in \mathrm{Dom}(g)$ and

$$|x - x_0| < \delta_g \quad \Longrightarrow \quad |g(x) - g(x_0)| < \delta_f.$$

In other words, for $\varepsilon > 0$ there is a $\delta = \delta_g > 0$ such that $x \in \mathrm{Dom}(g)$ and $|x - x_0| < \delta$ implies

$$|y - y_0| = |g(x) - g(x_0)| < \delta_f$$

Now if $y \in \mathrm{Dom}(f)$, then this last inequality implies

$$|f(g(x)) - f(g(x_0))| = |f(y) - f(y_0)| < \varepsilon$$

as desired.

□

Theorem 2. The Intermediate Value Theorem

Suppose that f is continuous on an interval I and $a, b \in I$ with $a < b$. Then f attains every value between $f(a)$ and $f(b)$. (*Note:* Functions that satisfy this property are said to satisfy the Intermediate Value Property (IVP).)

Proof. Suppose that L lies between $f(a)$ and $f(b)$. Without loss of generality, we may assume that $f(a) < L < f(b)$. The theorem asserts that there is a $c \in (a, b)$ such that $f(c) = L$.

Now let $A = \{x \in [a, b] : f(x) < L\}$. Clearly A is nonempty since $a \in A$. Also, A is certainly bounded above by b. So by the Axiom of Completeness, $c = \sup A < \infty$. We claim that $f(c) = L$ (and $c \in (a, b)$).

Observe that

(1) $$a \leq c \leq b \quad \text{(Why?)}$$

Now there are 3 possibilities. Either $f(c) < L, f(c) = L,$ or $f(c) > L$.

First suppose that $f(c) > L$. Then by Proposition 2 from section 2.17 (class notes), there is a $\delta > 0$ so that $f(x) > L$ for all $x \in (c - \delta, c + \delta)$. In particular,

$$c - \delta < x < c \quad \Longrightarrow \quad f(x) > L$$

In other words, $A \cap (c - \delta, c) = \emptyset$ and hence c is not the supremum of A. Notice that this last result also shows that $c \neq b$ in (1).

Now suppose that $f(c) < L$ and let $\varepsilon = L - f(c) > 0$. Then there is a $\delta > 0$ such that for all $x \in (c - \delta, c + \delta)$, we have

$$-\varepsilon < f(x) - f(c) < \varepsilon$$
$$f(c) - L < f(x) - f(c) < L - f(c)$$

Focusing on the right inequality, we see that there is $x_0 \in (c, b) \cap (c, c + \delta)$ such that

$$f(x_0) < L$$

But then c is not an upper bound of A. Finally, notice that this also shows that $c \neq a$ in (1).

It follows that $a < c < b$ and $f(c) = L$, as desired.

$\square$

The modern definition of continuity and an early version of the Intermediate Value Theorem (with $L = 0$ were discovered by Bernard Bolzano in the early part of the nineteenth century. It is an interesting fact that some of the mathematicians at the time attempted to define a continuous function as a function that satisfies the Intermediate Value Property (IVP). However, we have the following example.

Example 1. Let $f(x) = \sin(1/x)$. Then f has the IVP property but f is not continuous at the origin.

The IVT is important enough that we sketch another proof – one based on the Monotone Convergence Theorem (or the Nested Interval Property). Once again let we suppose that $a < b$ and that $f(a) < L < f(b)$ for some $L \in \mathbb{R}$. Let $a_0 = a$ and $b_0 = b$ and bisect the interval $[a_0, b_0]$ to obtain a midpoint M_0. As usual, there are two possibilities. Either $f(M_0) \leq L$ or $f(M_0) \geq L$. If the former, let $a_1 = a_0$ and $b_1 = M_0$. Otherwise, let $a_1 = M_0$ and $b_1 = b_0$. In either case, we have created a new (nested) interval $[a_1, b_1]$ such that

$$f(a_1) \leq L \leq f(b_1) \quad \text{and} \quad a_0 \leq a_1 < b_1 \leq b_0$$

Now continue the process to create a bounded increasing sequence $\{a_n\}$ and a bounded decreasing sequence $\{b_n\}$ with the property

(2)
$$f(a_n) \leq L \leq f(b_n) \quad \text{and} \quad a_{n-1} \leq a_n < b_n \leq b_{n-1}$$

for each $n \in \mathbb{N}$.

By the MCT, both sequences converge. In fact, they converge to a common limit, call it c (why?). Now use continuity and (2) to conclude that

$$f(c) = \lim_{n\to\infty} f(a_n) \leq L \leq \lim_{n\to\infty} f(b_n) = f(c)$$

The result follows. We leave it as an exercise to fill in the missing details.

Theorem 3. Max-Min Theorem

Let f be a continuous function defined on a closed bounded interval. Then f is bounded. In fact, f attains its minimum and maximum values somewhere on the interval. More precisely, suppose f is a continuous function defined on $I = [a, b]$ for some $a, b \in \mathbb{R}$. Then there exist $c, d \in I$ such that

(3) $$f(c) \leq f(x) \leq f(d) \quad \text{for all } x \in I$$

Note: Whenever (3) holds we say that f *attains* its minimum and maximum values.

Proof. We first show that there exists an $M > 0$ such that $|f(x)| \leq M$ for all $x \in [a, b]$. Equivalently, we need to show that the following set is bounded above.

$$A = \{|f(x)| : x \in [a, b]\}$$

To see this, let $y = |f(x)|$ and suppose A is not bounded above. Then we can find a sequence $\{y_n\} \subset A$ such $y_n \to \infty$ as $n \to \infty$. Now for each $n \in \mathbb{N}$, $y_n \in A$ so there is a corresponding $x_n \in [a, b]$ such that $y_n = |f(x_n)|$. Now the sequence $\{x_n\}$ is bounded since it is contained in $[a, b]$. So by the Bolzano-Weierstrass Theorem, there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = c \in [a, b]$. The continuity of $|f|$ implies that $\lim_{k\to\infty} |f(x_{n_k})| = |f(c)| \in \mathbb{R}$. In other words, we have found a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$. This is impossible. It follows that A is bounded.

Now we focus on the first inequality in (3). This time we dispense with the introduction of an intermediate variable (y_n above). So let

$$B = \{f(x) : x \in [a, b]\} \neq \emptyset$$

Since B is bounded, $\alpha = \inf B$ is finite. By the alternative characterization of the infimum, for each $n \in \mathbb{N}$, there is an element in B, call it $f(x_n)$, such that

$$\alpha \leq f(x_n) < \alpha + \frac{1}{n}$$

It follows that

$$f(x_n) \to \alpha \text{ as } n \to \infty.$$

Now observe that the sequence $\{x_n\}$ is bounded. Once again, by the Bolzano-Weierstrass Theorem, there is a subsequence $\{x_{n_k}\}$ such that $\lim_{k\to\infty} x_{n_k} = c$ for some $c \in [a, b]$. So by the continuity of f, we must have

$$f(c) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{n\to\infty} f(x_n) = \alpha = \inf B$$

That is, for all $x \in [a, b]$

$$f(c) \leq f(x)$$

The proof that f attains its maximum value is similar and is left as an exercise.

$\square$