3.19 Uniform Continuity

Definition. Uniform Continuity

Let f be defined on $D \subset \mathbb{R}$. We say that $f(x)$ is uniformly continuous on D, provided that for every $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that if $x, y \in D$ with

(1) $$|x - y| < \delta \quad \text{then} \quad |f(x) - f(y)| < \varepsilon$$

Remark. So uniform continuity is a *global* property. For each $\varepsilon > 0$ we must find a *single* $\delta > 0$, independent of x and y, so that (1) holds.

Example 1. Let $m, b \in \mathbb{R}$. Prove that $f(x) = mx + b$ is uniformly continuous on $\mathbb{R}$.

Proof. Given $\varepsilon > 0$, we must find a $\delta > 0$ so that (1) holds. We claim that $\delta = \varepsilon/(|m| + 1)$ will suffice. Now suppose that $|x - y| < \delta$. Then

$$\begin{aligned} |f(x) - f(y)| &= |(mx + b) - (my + b)| \\ &= |m|\,|x - y| \\ &< |m| \times \frac{\varepsilon}{|m| + 1} \\ &< \varepsilon \end{aligned}$$

□

It is easy to see that uniform continuity implies continuity since

$$|f(x) - f(y)| \le |f(x) - f(c)| + |f(c) - f(y)|$$

for any $c \in \mathrm{Dom}(f)$. However, the converse is not true. Consider the next example.

Example 2. Show that $f(x) = x^2$ is not uniformly continuous. Let $\delta > 0$ and choose $x > 1/\delta$. Now let $y = x + \delta/2$. Then $|x - y| = \delta/2 < \delta$, however,

$$\begin{aligned} |f(x) - f(y)| &= |x^2 - y^2| = |x + y|\,|x - y| \\ &= (x + y)\frac{\delta}{2} > 2x\,\frac{\delta}{2} \\ &> \frac{2}{\delta}\,\frac{\delta}{2} = 1 \end{aligned}$$

Rather the sketch the graph of $z = f(x + \delta/2) - f(x)$ in 3-dimensions, we will sketch a few graphs for a given value of (the parameter) δ in 2-dimensions. So let

$$F_\delta(x) = f(x + \delta/2) - f(x)$$

Figure 1 shows the graphs of $y = F_\delta(x)$ for $\delta = 1/2, 1/5, 1/10$.

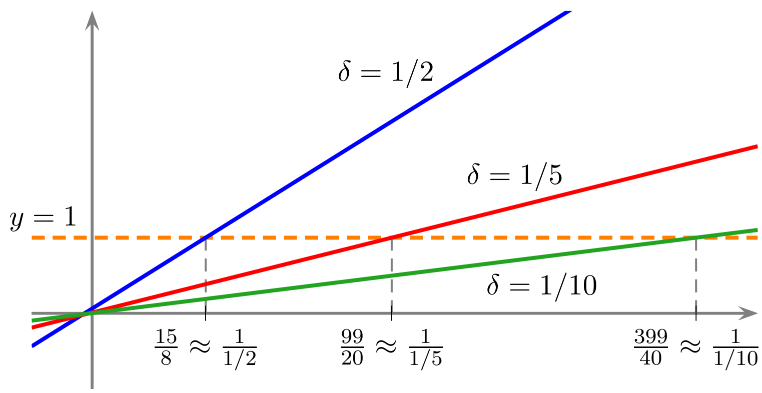


Figure 1: The Graphs of $y = F_\delta(x)$

So the general idea is this: If f were absolutely continuous, then if we chose $\delta > 0$ small enough, then $|f(x) - f(y)|$ could be made small as long as $|x - y| < \delta$. However, the sketch makes clear that this won't work. For example, if $\delta = 1/5$, then

$$x > 1/\delta \quad \text{implies} \quad f\left(x + \frac{1/5}{2}\right) - f(x) > 1.$$

It is possible that continuity plus an additional condition will guarantee uniform continuity. Although not completely necessary, such a discussion is enhanced by a deeper understanding of the structure of subsets of the real line.

Point Set Topology

Definition. A set $U \subset \mathbb{R}$ is called **open** if for every $x \in U$ there is a $\delta > 0$ such that $(x-\delta, x+\delta) \subset U$. A set is called **closed** if it is the compliment of an open set. That is, $F \subset \mathbb{R}$ is closed if $F^c = \mathbb{R} \setminus F$ is an open set.

Theorem 1. Let A and B be open sets in $\mathbb{R}$. Then $A \cup B$ and $A \cap B$ are open.

Theorem 2. Let F and G be closed sets in $\mathbb{R}$. Then $F \cup G$ and $F \cap G$ are also closed.

Example 3. Let $a, b \in \mathbb{R}$.

a. $\mathbb{R}$ and $\emptyset$ are both open and closed.

b. $(a,b), (-\infty, b),$ and (a, ∞) are examples of open sets.

c. $[a,b], (-\infty, b],$ and $[a, \infty)$ are examples of closed sets.

d. The singleton set $\{a\}$ is closed.

The following examples show that Theorems 1 and 2 can only be partially extended.

Example 4.

a. $\displaystyle\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, \frac{n}{n+1}\right] = (0,1)$

b. $\displaystyle\bigcap_{n=1}^{\infty} \left(\frac{-1}{n}, \frac{n+1}{n}\right) = [0,1]$

However, we do have

Theorem 3. Let $\{U_n\}$ be a sequence of open sets. Then

a. $\bigcup_{j=1}^{\infty} U_j$ is open.

b. For each $n \in \mathbb{N}$, $\bigcap_{j=1}^{n} U_j$ is open.

Theorem 4. Let $\{F_n\}$ be a sequence of closed sets. Then

a. $\bigcap_{j=1}^{\infty} F_j$ is closed.

b. For each $n \in \mathbb{N}$, $\bigcup_{j=1}^{n} F_j$ is closed.

Theorem 5. Let $F \subset \mathbb{R}$. Then F is closed if and only F contains all of its limit points.

Definition. Let $A \subset \mathbb{R}$. A collection Γ of *open* sets is called an **open cover** of A if

$$A \subset \bigcup_{O \in \Gamma} O$$

Theorem 6. Let $A = [a, b]$ be a closed bounded interval and let Γ be an open cover of A. Then there is a finite number of elements of Γ that cover A. That is,

$$A \subset \bigcup_{j=1}^{n} U_j$$

for some finite subset $\{U_1, U_2, \ldots, U_n\}$ of elements from Γ.

Note: The above statement is usually shortened to "Let A be a closed bounded interval. Then every open cover of A has a finite subcover".

Proof. Let Γ be an open cover of A and define

$$E = \{x \in [a, b] : [a, x] \text{ has a finite subcover}\}$$

E is nonempty since $a \in E$. E is also bounded above by b, so $a \leq c = \sup E \leq b$. First we claim that $c \in E$. If the claim is true, then there is a finite subset $\{U_1, U_2, \ldots, U_n\} \subset \Gamma$ such that $[a, c] \subset \bigcup_{j=1}^{n} U_j$.

Now suppose that $c < b$ and choose a element $U \in \Gamma$ with $c \in U$. Since U is an open set, there is an $\varepsilon_1 > 0$ such that $(c - \varepsilon_1, c + \varepsilon_1) \subset U$. Let $\varepsilon = \min\{\varepsilon_1, b - c\}$. Then $c < c + \varepsilon/2 < b$ and $\{U_1, U_2, \ldots, U_n, U\}$ is a finite subcover of $[a, c + \varepsilon/2]$ contrary to our choice of c. It follows that $c = b$.

To complete the proof we must show that $c \in E$. Since Γ is an open cover of A (and since $c \in [a, b]$), there is an element in $V \in \Gamma$ such that $c \in V$. Since V is open there is an $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subset V$. And since $c = \sup E$, there is an $x_0 \in E$ such that $c - \varepsilon < x_0 \leq c$.

Now $x_0 \in E$ implies that $[a, x_0] \subset \bigcup_{j=1}^n U_j$ for some finite subset $\{U_1, U_2, \ldots, U_n\} \subset \Gamma$. Also, $[x_0, c] \subset V$. Thus

(2) $$[a, c] \subset \left(V \cup \bigcup_{j=1}^{n} U_j \right)$$

In other words, $c \in E$.

$\square$

Definition. A set A is called **compact** if every open cover of A has a finite subcover.

Theorem 6 says that every closed bounded interval is compact. Actually, we can say more.

Theorem 7. Heine-Borel

A subset of $\mathbb{R}$ is compact if and only if it is closed and bounded.

Remark. The theorem actually holds in higher dimensions (i.e., $\mathbb{R}^n$). However, there are metric spaces that contain closed and bounded sets that are not compact. Notice also that Theorem 7 is more general since it is not restricted to closed intervals, but holds for arbitrary closed (and bounded) sets.

Example 2 was disappointing. However, we have the following partial converse.

Theorem 8. Let K be a compact subset of the real numbers and suppose that $f : K \longrightarrow \mathbb{R}$ is a continuous function. Then f is uniformly continuous on K.

Proof. Let $\varepsilon > 0$. By the continuity of f, there exists $\delta_x = \delta(x, \varepsilon) > 0$ such that

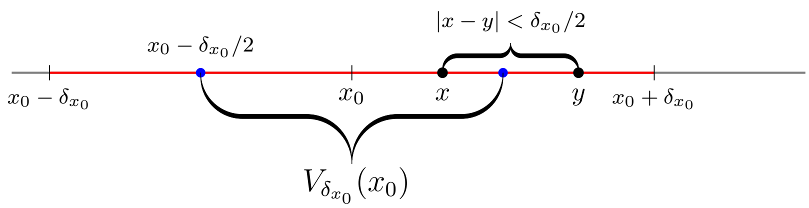
(3) $$|y - x| < \delta_x \quad \Longrightarrow \quad |f(y) - f(x)| < \varepsilon/2$$

This holds for each $x \in K$.

Now let $V_{\delta_x}(x) = (x - \delta_x/2, x + \delta_x/2)$. Then the collection $\Gamma = \{V_{\delta_x}(x) : x \in K\}$ is an open cover of K since

$$K \subset \bigcup_{x \in K} V_{\delta_x}(x)$$

The justification for choosing a radius of $\delta/2$ for each the intervals is as follows. Looking ahead our plan is to invoke the Heine-Borel Theorem to obtain a finite subcollection from Γ. If we halve the size of each interval ahead of time, then keeping x and y sufficiently close should force both to belong to the same interval. For example, if $x \in (x_0 - \delta_{x_0}/2, x_0 + \delta_{x_0}/2)$ then $y \in (x_0 - \delta_{x_0}, x_0 + \delta_{x_0})$ as shown below.



Since K is compact, Theorem 7 implies that there is a finite subcollection of Γ, say $\{V_{\delta_1}(x_1), V_{\delta_2}(x_2), \ldots, V_{\delta_n}(x_n)\}$, such that

$$K \subset \bigcup_{j=1}^{n} V_{\delta_j}(x_j)$$

Let $\delta = \frac{1}{2}\min\{\delta_1, \delta_2, \ldots, \delta_n\}$ and let $x, y \in K$ with $|y-x| < \delta$. Now $x \in V_{\delta_j}(x_j)$ for some $1 \le j \le n$. But then

$$\begin{aligned} |y-x_j| &\le |y-x| + |x-x_j| \\ &< \delta + \delta_j/2 \\ &\le \delta_j/2 + \delta_j/2 = \delta_j \end{aligned}$$

Hence by (3),

$$|f(y) - f(x_j)| < \varepsilon/2$$

It follows that

$$\begin{aligned} |f(y)-f(x)| &= |f(y)-f(x_j)+f(x_j)-f(x)| \\ &\le |f(y)-f(x_j)| + |f(x_j)-f(x)| \\ &< \varepsilon/2 + \varepsilon/2 \end{aligned}$$

□

We finish this section by presenting another proof of Theorem 8.

Proof. Let $\varepsilon > 0$. If f is not uniformly continuous, then for each $n \in \mathbb{N}$ we can find a pair of points $x_n, y_n \in K$ such that $|x_n - y_n| < 1/n$ but

(4) $$|f(x_n) - f(y_n)| \geq \varepsilon.$$

So we have constructed a pair of sequences $\{x_n\}$ and $\{y_n\}$ with the property that $\lim_{n\to\infty}(x_n - y_n) = 0$.

Since K is compact, the Heine-Borel Theorem tells us that it is closed and bounded. Hence $\{x_n\} \subset K$ is a bounded sequence and by the Bolzano-Weierstrass Theorem, has a convergent subsequence $\{x_{n_k}\}$, with $\lim_{k\to\infty} x_{n_k} = c$. It follows that c is a limit point of K, and since K is closed, we must have $c \in K$.

Clearly,

$$\begin{aligned}\lim_{k\to\infty} y_{n_k} &= \lim_{k\to\infty}(y_{n_k} - x_{n_k} + x_{n_k}) \\ &= \lim_{k\to\infty}(y_{n_k} - x_{n_k}) + \lim_{k\to\infty} x_{n_k} \\ &= 0 + \lim_{k\to\infty} x_{n_k} = c\end{aligned}$$

Now by the continuity of f at c and (4), we have

$$\begin{aligned}0 &= |f(c) - f(c)| \\ &= |\lim_{k\to\infty} f(x_{n_k}) - \lim_{k\to\infty} f(y_{n_k})| \\ &= \lim_{k\to\infty} |f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon\end{aligned}$$

This is absurd. We conclude that f is uniformly continuous on K.

□