3.20 One-Sided Limits

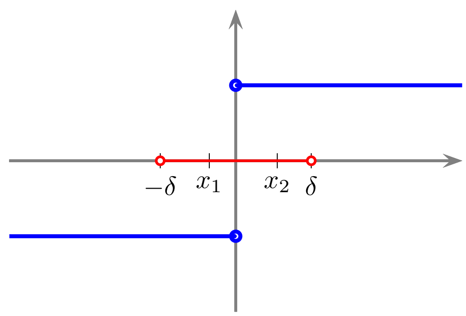
The first part of this section is motivated by the following example.

Example 1. Find the limit below or show that it does not exist.

(1) $$\lim_{x\to 0} \frac{|x|}{x}$$

A sketch of the graph of $y = f(x) = |x|/x$ suggests that this limit does not exist. To see this let $\epsilon = 1/2$. Notice that for every choice of $\delta > 0$ we can find $x_1, x_2 \in (-\delta, \delta)$ such that

$$|f(x_2) - f(x_1)| = |1 - (-1)| = 2 > 1/2 = \epsilon$$



How can we make this more precise?

Left and Right-Hand Limits

Definition. One-Sided Limits

Let $f(x)$ be defined on the open interval (c, b). We say that the limit as x approaches c **from the right** of $f(x)$ is the real number L and we write

$$\lim_{x \to c^+} f(x) = L$$

provided that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

(2) $$0 < x - c < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

Similarly, let $f(x)$ be defined on the open interval (a, c). We say that the limit as x approaches c **from the left** of $f(x)$ is the real number L and we write

$$\lim_{x \to c^-} f(x) = L$$

provided that for every $\varepsilon > 0$ there is a $\delta > 0$ such that

(3) $$0 < c - x < \delta \quad \Longrightarrow \quad |f(x) - L| < \varepsilon$$

Theorem 1.

$$\lim_{x\to c} f(x) = L \qquad \Longleftrightarrow \qquad \lim_{x\to c^-} f(x) = \lim_{x\to c^+} f(x) = L$$

Remark. This theorem is used primarily to show that certain limits *do not exist*.

Example 1. (cont.)

Find the limit below or show that it does not exist.

$$\lim_{x\to 0} \frac{|x|}{x}$$

We claim the limit does not exist. To see this, observe that

$$\lim_{x\to 0^-} \frac{|x|}{x} = \underbrace{\lim_{x\to 0^-}}_{\Longrightarrow x<0} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1$$

$$\lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} 1 = 1$$

In other words, the left-hand limit does not equal the right-hand limit.

The Limit as θ approaches 0 of $\sin\theta/\theta$

The following fact is needed below.

Proposition 2.

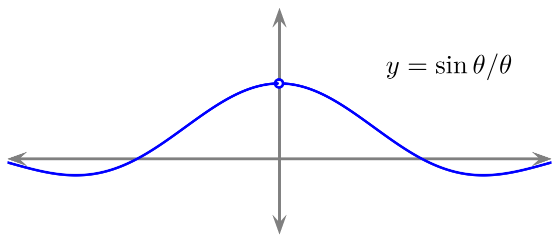
(4)
$$\lim_{x\to 0}\cos x = 1$$

For a proof, consult almost any first year calculus text.

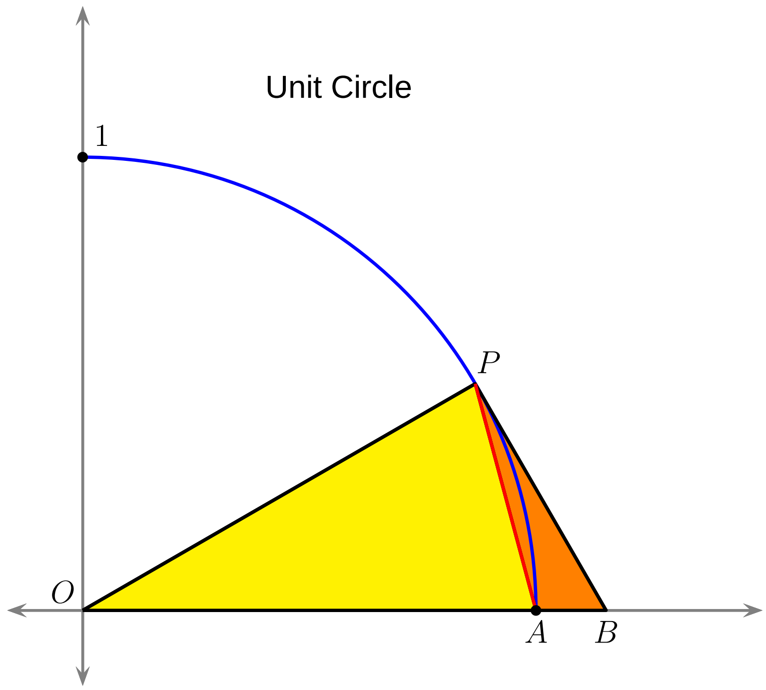
Notice that the (indeterminate) limit below is of the form $0/0$. Unfortunately, there is no algebraic simplification that we can use to eliminate θ from the denominator.

(5)
$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} =$$

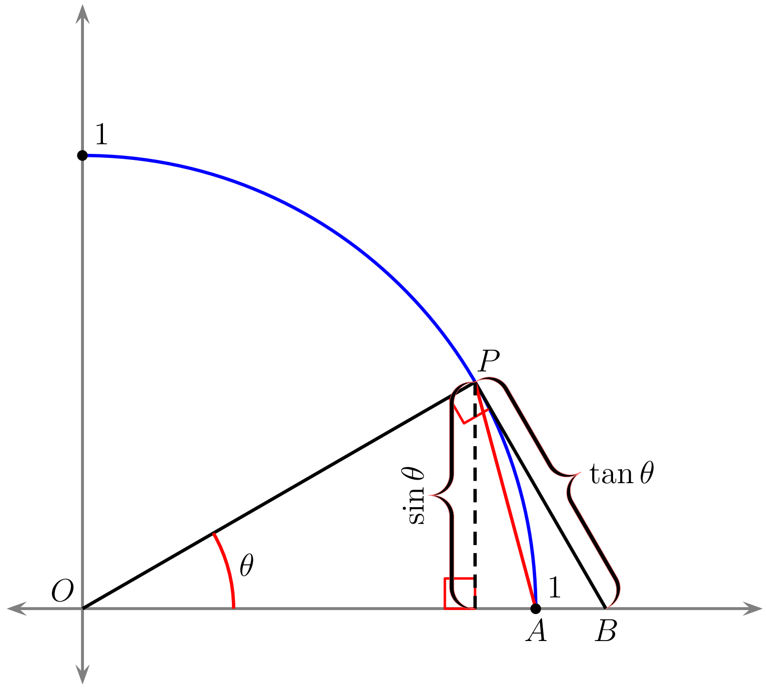
The sketch below suggests that the limit does indeed exist and is equal to 1.



How do we go about proving such a result? The following sketches suggest a possible approach.



Notice that area $\triangle OAP <$ area of sector $OAP <$ area $\triangle OBP$



(A complete summary relating the trigonometric functions using the unit circle is included at the end of this section.)

Thus if $\pi/2 > \theta > 0$ then

$$\frac{1}{2}\sin\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta$$

The first inequality implies that

$$\frac{\sin\theta}{\theta} < 1.$$

The second implies

$$\frac{\tan\theta}{\theta} > 1$$

or

$$\frac{\sin\theta}{\theta} > \cos\theta.$$

Thus

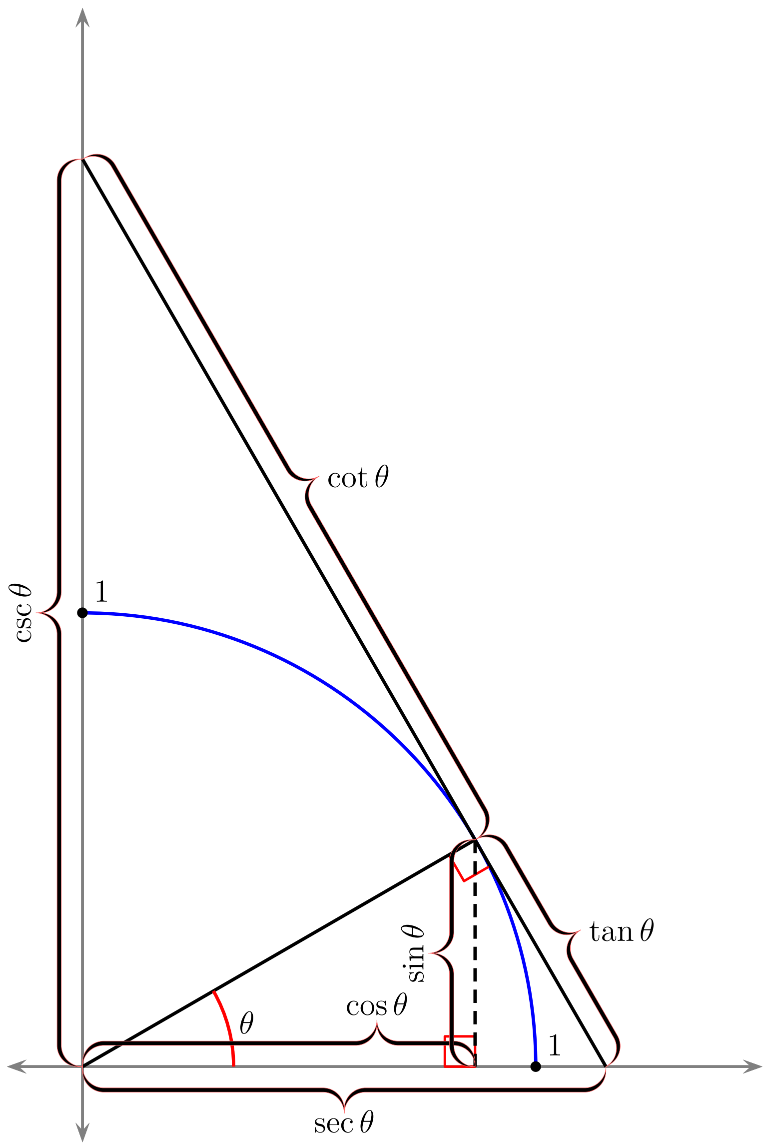
$$\cos\theta < \frac{\sin\theta}{\theta} < 1$$

Now we apply Proposition 2 and the Squeeze Law to obtain

Theorem 3.

(6)
$$\lim_{x \to 0} \frac{\sin x}{x} = 1$$

The sketch below is easily constructed using the unit circle and elementary trigonometry.



The blue arc is part of the unit circle.