4.23 Power Series

Recall the geometric series

(1) $$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots$$

As we saw earlier, the series (1) diverges if the common ratio $|x| > 1$ and converges if $|x| < 1$. In fact, for all $x \in (-1, 1)$ this series has the "closed form" representation

(2) $$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad -1 < x < 1$$

Also, the series is clearly divergent if $x = 1$ since

$$1 + 1 + \cdots + 1 + \cdots = \infty$$

Finally, for $x = -1$ we have

(3) $$1 - 1 + 1 - 1 + \cdots + (-1)^{n+1} + \cdots$$

which is also divergent since the terms do not approach 0. We'll return to this case later.

Power Series and Convergence

Equation (1) is an example of a power series. Formally, we have

Definition. Power Series, Center, and Coefficients
A **power series about** $x = a$ is a series of the form

(4) $$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

The **center** a and **coefficients** $c_0, c_1, \ldots, c_n, \ldots$ are constants.

Remark.

(i) For many examples the center is chosen to be 0.

(ii) Notice that every power series converges (trivially) at its center. The question is, “for what other x-values does the series in (4) converge?”.

For example, the series in (1) is a power series centered at $x = 0$ and the coefficients are $c_0 = 1, c_1 = 1, \ldots, c_n = 1, \ldots$. That is,

$$\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} 1 \cdot (x-0)^n$$

Example 1. Testing for Convergence
For which values of x does the following series converge?

$$\sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

Notice that the center is 0 and the coefficients are $c_n = 2^n$. We try the Ratio Test (Actually, the Root Test is a better choice here!). Let $a_n = (2x)^n$. Then

$$\begin{aligned}\rho &= \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| \\ &= \lim_{n\to\infty}\left|\frac{(2x)^{n+1}}{(2x)^n}\right| \\ &= |2x|\end{aligned}$$

It follows that the series converges absolutely if

$$|2x| < 1$$

Notice that, by the Ratio Test, this series diverges for all $|x| > 1/2$. In general, the end points must be always be explicitly checked. In this example, the series also diverges at $\pm 1/2$ as one can easily verify.

The interval $(-1/2, 1/2)$ is called the **interval of convergence**.

Example 2. Testing for Convergence (cont.)
For which values of x does the following series converge?

$$\sum_{n=1}^{\infty} \frac{(3x-5)^n}{\sqrt{n}}$$

In this example the center is $a = 5/3$ and $c_n = 3^n/\sqrt{n}$. Again we try the Ratio Test. Let $a_n = (3x-5)^n/\sqrt{n}$. Then

$$\begin{aligned} \rho &= \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| \\ &= \lim_{n\to\infty} \left| \frac{(3x-5)^{n+1}}{\sqrt{n+1}} \frac{\sqrt{n}}{(3x-5)^n} \right| \\ &= |3x-5| \end{aligned}$$

It follows that the series converges absolutely if $\rho < 1$, that is if

$$\begin{aligned} -1 < 3x - 5 < 1 \\ \Longrightarrow x \in (4/3, 2) \text{ or} \\ x \in (a - 1/3, a + 1/3) = I \end{aligned}$$

Remark. Once again, I is called the interval of convergence. Also, the number, $1/3$, is called the **radius of convergence**. It is not difficult to verify the the series diverges for $x \geq 2$ and $x < 4/3$. Also, notice that the series converges conditionally at $x = 4/3$ by Leibniz's Theorem.

Theorem 1. The Convergence Theorem for Power Series
If the power series $\sum a_n x^n$ converges for $x = c \neq 0$, then the series converges absolutely for all x with $|x| < |c|$. If the series diverges for some $x = d$, then it diverges for all x with $|x| > d$.

Corollary 2. Corollary to Theorem 1
The convergence of the series $\sum a_n (x-a)^n$ has only one of three possibilities.

(i) There is a positive number R (called the **radius of convergence**) such that the series diverges for all x with $|x-a| > R$ but converges absolutely for all x with $|x-a| < R$. The series must be explicitly tested at the end points $x = a \pm R$.

(ii) The series converges absolutely for all x. (In this case, $R = \infty$.)

(iii) The series converges at $x = a$ only and diverges elsewhere. (In this case, $R = 0$.)

Remark. If $\sum c_n (x-a)^n$ converges for $x \in (a-R, a+R),\ R > 0$ then the power series defines a function f:

(5)
$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n, \quad a - R < x < a + R$$

Example 3. Geometric Series
Earlier we saw that the series $\sum_{n=0}^{\infty} x^n$ converged absolutely on the interval $(-1,1)$. So for all $x \in (-1,1)$ this series defines a function, say f. We have

$$f(x) = \sum_{n=0}^{\infty} x^n, \quad -1 < x < 1$$

In fact, $f(x)$ has the "closed" form.

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1$$

Example 4. Find the interval of convergence for the power series

$$1 - x + x^2 - x^3 + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \tag{6}$$

It is easy to see (by the ratio test) that the series in (6) has the same interval of convergence as the series in the previous example. Also

$$\begin{aligned} \frac{1}{1+x} &= \frac{1}{1-(-x)} \\ &= 1 - x + x^2 - x^3 + x^4 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n x^n \end{aligned}$$

for $x \in (-1,1)$. It follows that the series in (6) defines a function g with

$$g(x) = \sum_{n=0}^{\infty} (-1)^n x^n = \frac{1}{1+x}, \quad x \in (-1,1) \tag{7}$$

It is worth noting that

$$\lim_{x \to 1^-} g(x) = \lim_{x \to 1^-} \frac{1}{1+x} = 1/2 \neq \sum_{n=0}^{\infty} (-1)^n$$

Theorem 3. Term-by-Term Differentiation Theorem

Suppose that (5) holds. That is, suppose

$$f(x) = \sum_{n=0}^{\infty} c_n\,(x-a)^n, \qquad a - R < x < a + R$$

Then f has derivatives of all orders inside the interval of convergence. In fact, we differentiate term-by-term. That is,

$$f'(x) = \sum_{n=1}^{\infty} n \cdot c_n\,(x-a)^{n-1}$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) \cdot c_n\,(x-a)^{n-2},$$

and so on. Each of the derived series converging at each point in $(a-R, a+R)$.

Example 5. Find the power series expansion of each of the following about $a = 0$. What is the interval of convergence?

(a) $\dfrac{1}{1+x^2}$

(b) $\dfrac{x}{(1+x^2)^2}$

As we saw in Example 4,

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 + \cdots$$

The substitution $x \to x^2$ to gives

$$\begin{aligned}\frac{1}{1+x^2} &= 1 - x^2 + (x^2)^2 - (x^2)^3 + (x^2)^4 + \cdots \\ &= 1 - x^2 + x^4 - x^6 + x^8 + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} x^{2n}\end{aligned}$$

and this series converges for all $-1 < x^2 < 1$. That is, for $-1 < x < 1$.

For part (b) we let $f(x) = 1/(1+x^2)$. Then

$$f'(x) = \frac{-2x}{(1+x^2)^2}$$

So by part (a)

$$\begin{aligned}\frac{x}{(1+x^2)^2} &= \frac{-1}{2}\, f'(x) \\ &= \frac{-1}{2}\,\frac{d}{dx}\,\left(1 - x^2 + x^4 - x^6 + x^8 + \cdots\right) \\ &= \frac{-1}{2}\left(0 - 2x + 4x^3 - 6x^5 + 8x^7 + \cdots\right)\end{aligned}$$

by Theorem 3. Since the power series in (a) converges for all $-1 < x < 1$, the series in (b) must have the same interval of convergence.

Example 6. Let

(8) $$h(x) = \sum_{n=1}^{\infty} (-1)^{n+1} n x^n$$

(a) Find the radius and interval of convergence. In other words, find the domain of h

It follows from a straight-forward application of the ratio test that the series converges (absolutely) for all $|x| < 1$. Hence the series in (8) defines a differentiable function on $(-1, 1)$.

(b) Show that $\lim_{x\to 1^-} h(x) = 1/4$.

Let

$$\begin{aligned}g(x) &= \sum_{n=0}^{\infty} (-1)^{n+1} x^n \\ &= -1 + x - x^2 + x^3 + \cdots \\ &= \frac{-1}{1+x}, \quad -1 < x < 1\end{aligned}$$

So by Theorem 3

$$\begin{aligned} g'(x) &= 1 - 2x + 3x^2 - 4x^3 + \cdots \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \\ &= \frac{1}{(1+x)^2}, \quad -1 < x < 1 \end{aligned}$$

Now observe that

$$\begin{aligned} h(x) &= \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n} \\ &= x \sum_{n=1}^{\infty} (-1)^{n+1} n x^{n-1} \\ &= x g'(x) \\ &= \frac{x}{(1+x)^2}, \quad -1 < x < 1 \end{aligned}$$

It follows that

(9) $$\lim_{x \to 1^-} h(x) = \lim_{x \to 1^-} \frac{x}{(1+x)^2} = 1/4, \qquad \text{(Why?)}$$

Remark. In section 2.15 we noted that the alternating series

$$\sum_{n=1}^{\infty}(-1)^{n+1}n = 1 - 2 + 3 - 4 + \cdots$$

diverged by the n^{th} term test. It follows that

$$h(1) = \sum_{n=1}^{\infty}(-1)^{n+1}n(1)^n = \sum_{n=1}^{\infty}(-1)^{n+1}n$$

does not exist. So h is not (left) continuous at $x = 1$ even though it has a (left-hand) limit there.

To elaborate further, let $k(x) = x/(1+x)^2$. Then $k(x)$ is defined for all real $x \neq -1$ but, the function $h(x)$, given in (8), is defined only for $x \in (-1, 1)$. In particular, $h(x) \neq k(x)$.

On the other hand, if we restrict ourselves to $x \in (-1, 1)$, then the two functions are equal. We used this fact to evaluate the limit in (9).

Summability Theory

The previous example touches on a subject called **Summability Theory**. A series $\sum_{n=0}^{\infty} a_n$ is said to be (Abel) summable (to L) if

(a) The power series $\sum_{n=0}^{\infty} a_n x^n$ converges for all $|x| < 1$ and,

(b) $f(x) = \sum_{n=0}^{\infty} a_n x^n \to L$ as $x \to 1^-$.

In the last example we showed that the *divergent* series $\sum(-1)^{n+1}n$ is Abel summable to $1/4$.

The fact the *convergent* series are necessarily (Abel) summable was proven by N. H. Abel in 1826.

Theorem 4. (**Abel**)
Suppose that $\sum_{n=0}^{\infty} a_n$ converges to a real number, say L. Then the (power) series $\sum_{n=0}^{\infty} a_n\, x^n$ converges for all $x \in (-1,1)$ and

$$\lim_{x\to 1^-} \sum_{n=0}^{\infty} a_n\, x^n = \sum_{n=0}^{\infty} a_n = L \tag{10}$$

Before proceeding with the proof, it is useful to rewrite the Abel sum in a more convenient form. As usual, let $s_n = \sum_{j=0}^{n} a_j$. Observe that

$$\begin{aligned}
\sum_{n=0}^{\infty} a_n\, x^n &= a_0 + \sum_{n=1}^{\infty} a_n\, x^n \\
&= s_0 + \sum_{n=1}^{\infty} \underbrace{(s_n - s_{n-1})}_{a_n}\, x^n \\
&= s_0 + \sum_{n=1}^{\infty} s_n\, x^n - \sum_{n=1}^{\infty} s_{n-1}\, x^n \\
&= s_0 + \sum_{n=1}^{\infty} s_n\, x^n - \sum_{n=0}^{\infty} s_n\, x^{n+1} \\
&= \sum_{n=0}^{\infty} s_n\, x^n - \sum_{n=0}^{\infty} s_n\, x^{n+1} \\
&= \sum_{n=0}^{\infty} s_n\, (x^n - x^{n+1}) \\
&= (1-x) \sum_{n=0}^{\infty} s_n\, x^n
\end{aligned}$$

Proof. We leave it as an exercise to show that

$$f(x) = \sum_{n=0}^{\infty} a_n \, x^n \tag{11}$$

converges for all $x \in (-1, 1)$.

To prove (ii) above, we proceed in much the same way as we did in the proof of Cesàro's Theorem (Section 2.14-15).

Let $\varepsilon > 0$. Now choose N so large that $|s_n - L| < \varepsilon$ whenever $n \geq N$. Then

$$\begin{aligned}
|f(x) - L| &= \left|(1-x)\sum_{n=0}^{\infty} s_n \, x^n - L\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty} s_n \, x^n - L(1-x)\sum_{n=0}^{\infty} x^n\right| \\
&= \left|(1-x)\sum_{n=0}^{\infty} s_n \, x^n - (1-x)\sum_{n=0}^{\infty} L x^n\right| \\
&\leq (1-x)\sum_{n=0}^{\infty} |s_n - L| \, x^n \\
&= (1-x)\sum_{n=0}^{N-1} \underbrace{|s_n - L|}_{\text{bounded}} \, x^n + (1-x)\sum_{n=N}^{\infty} \underbrace{|s_n - L|}_{<\varepsilon} \, x^n \\
&< (1-x)K\sum_{n=0}^{N-1} x^n + (1-x)\varepsilon\sum_{n=N}^{\infty} x^n \\
&< (1-x)K\sum_{n=0}^{N-1} 1 + (1-x)\varepsilon\sum_{n=0}^{\infty} x^n \\
&= (1-x)K\,N + (1-x)\frac{\varepsilon}{1-x} \\
&= (1-x)K\,N + \varepsilon
\end{aligned}$$

Now let $x \to 1^-$ and the result follows. □

It is interesting to compare the Abel sum with the Cesàro sum (from Section 2.14-15). Let $p(j) = 1 - j/n,\ j = 0, 1, 2 \ldots n-1$. Given a (formal) series $\sum_{n=0}^{\infty} a_n$, its Cesàro sum is defined by

$$\begin{aligned}
\sigma_n &= \sum_{j=0}^{n-1} \left(1 - \frac{j}{n}\right) a_j \\
&= a_0\, p(0) + \sum_{j=1}^{n-1} a_j\, p(j) \\
&= s_0\, p(0) + \sum_{j=1}^{n-1} (s_j - s_{j-1})\, p(j) \\
&= s_0\, p(0) + \sum_{j=1}^{n-1} s_j\, p(j) - \sum_{j=1}^{n-1} s_{j-1}\, p(j) \\
&= \sum_{j=0}^{n-1} s_j\, p(j) - \sum_{j=0}^{n-1} s_j\, p(j+1), \quad \text{(since } p(n) = 0) \\
&= \sum_{j=0}^{n-1} s_j\, (p(j) - p(j+1)) \\
&= \sum_{j=0}^{n-1} s_j\, \left(1 - \frac{j}{n} - 1 + \frac{j+1}{n}\right) \\
&= \frac{1}{n} \sum_{j=0}^{n-1} s_j \\
&= \frac{1}{\sum_{j=0}^{n-1} 1} \sum_{j=0}^{n-1} (s_j \times 1)
\end{aligned}$$

And its Abel sum is given by

$$\begin{aligned}\sum_{n=0}^{\infty} a_n\, x^n &= (1-x)\sum_{n=0}^{\infty} s_n\, x^n \\ &= \frac{1}{\sum_{n=0}^{\infty} x^n}\, \sum_{n=0}^{\infty} s_n\, x^n\end{aligned}$$

Comparing the final form of both sums, we see that Cesàro and Abel sums represent a sort of “averaging” process.

Theorem 5. Term-by-Term Integration Theorem
Suppose that (5) holds. That is, suppose

$$f(x) = \sum_{n=0}^{\infty} c_n\,(x-a)^n$$

converges for $a - R < x < a + R$. Then

$$\sum_{n=0}^{\infty} c_n\, \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\, dx = \sum_{n=0}^{\infty} c_n\, \frac{(x-a)^{n+1}}{n+1}$$

for $a - R < x < a + R$

Example 7. Find the power series expansion of $f(x) = \ln(1+x)$. Also, find the interval of convergence.
Observe that

$$\begin{aligned}f'(x) &= \frac{1}{1+x} \\ &= 1 - x + x^2 - x^3 + \cdots\,, \quad -1 < x < 1\end{aligned}$$

It follows by Theorem 5 that

$$\begin{aligned}f(x) &= \ln(1+x) \\ &= C + x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\,, \quad -1 < x < 1\end{aligned}$$

The initial condition $f(0) = \ln 1 = 0 \Longrightarrow C = 0$. Hence

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}, \quad -1 < x < 1$$

Recall that the Alternating Harmonic Series converges. Call its limit L. Now we can apply Abel's Theorem (Theorem 4) to conclude that

$$L = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \lim_{x \to 1^-} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \lim_{x \to 1^-} \ln(1+x) = \ln 2$$

Example 8. Find the power series expansion of

$$\int \frac{\sin x}{x}\, dx$$

about $x = 0$.
We know from Calculus that the power series expansion for the sine function about $x = 0$ is

$$\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \quad x \in \mathbb{R} \end{aligned}$$

It follows that

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots + (-1)^n \frac{x^{2n}}{(2n+1)!} + \cdots$$

So by Theorem 5,

$$\int \frac{\sin x}{x}\, dx = x - \frac{x^3}{3(3!)} - \frac{x^5}{5(5!)} + \frac{x^7}{7(7!)} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)(2n+1)!} + \cdots$$

for $n = 0, 1, 2, \ldots$ and the series converges for all $x \in \mathbb{R}$.